# CHAPTER 11: INFERENCES ON TWO SAMPLES

[This chapter is based on Chapter 11 of the textbook]

In Chapters 9 and 10 we discussed inference methods about a specific sample, and how to use the sample to draw conclusions about a population parameter such as the mean and proportion. In this chapter we will learn inferential methods for comparing two independent populations. These methods can be extended to dependent samples, but we won't do that here.

Before we start discussing the methods, let's define independent and dependent samples.

**Definition 11.1.**

*(i) A sampling method is independent when an individual selected for one sample does not dictate which individual is to be in the second sample.*

*(ii) A sampling method is dependent when an individual selected to be in one sample is used to determine the individual in the second sample.*

Let's see some examples.

**Example 11.1.** *Determine if the sampling method is independent or dependent in the following situations.*

*(a) Among competing acne medications, does one perform better than the other? To answer this question, researchers applied Medication A to one part of the subject's face and Medication B to a different part of the subject's face to determine the proportion of subjects whose acne cleared up for each medication. The part of the face that received Medication A was randomly determined.*

*(b) Do individuals who make fast-food purchases with a credit card tend to spend more than those who pay with cash? To answer this question, a marketing manager randomly selects 30 credit-card receipts and 30 cash receipts to determine if the credit-card receipts have a significantly higher dollar amount, on average.*

**Solution.**

(a) The same individuals are used to test both medicines. Then, sample $A$ and sample $B$ are composed of the same people. Therefore, the sampling method is dependent.

(b) Two groups of people are analyzed, and there is no overlap between the samples. Hence, the sampling method is independent.

□

## 11.1 Test hypotheses regarding two proportions from independent samples

We use our knowledge about the sampling distribution of $\hat{p}$ and extend it to two samples. If the first sample has a sample proportion $\hat{p}_1$ and the second sample has sample proportion $\hat{p}_2$. Then, both come from the same population if the corresponding population parameters $p_1$ and $p_2$ match. Equivalently, we want to test the null hypothesis

$$H_0 : p_1 - p_2 = 0$$

vs one of the following alternative hypotheses:

(i) Two-tailed: $H_1 : p_1 - p_2 \neq 0$

(ii) Left-tailed: $H_1 : p_1 - p_2 < 0$

(iii) Right-tailed: $H_1 : p_1 > p_2$.

Our hypothesis test for the population proportion with one sample was based on the idea that $\hat{p}$ is normally distributed with mean $\mu_{\hat{p}} = p$ and standard deviation $\sigma_{\hat{p}} = \sqrt{\frac{p(1-p}{n}}$. For two samples, our test is based on the idea that $\hat{p}_1 - \hat{p}_2$ is also normally distributed, with mean and standard deviation:

$$\mu_{\hat{p}_1-\hat{p}_2} = p_1 - p_2 \qquad \text{and} \qquad \sigma_{\hat{p}_1-\hat{p}_2} = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

where $n_1$ is the number of individuals in the first sample, and $n_2$ in the second sample. Observe that, assuming that the null hypothesis is true, we have $\mu_{\hat{p}_1-\hat{p}_2} = 0$. Then, we obtain the following hypothesis testing method:

(0) Verify conditions:

- The sampling method is independent
- The sample size is no more than 5% of the corresponding population. If $n_1$ is the sample size of the first sample, $N_1$ the population size of the first population, $n_2$ the sample size of the second sample and $N_2$ the population size of the second population, then we need

$$n_1 \leq 5\% N_1 \qquad \text{and} \qquad n_2 \leq 5\% N_2$$

- The sample sizes and proportions satisfy:

$$n_1\hat{p}_1(1-\hat{p}_1) \geq 10 \qquad \text{and} \qquad n_2\hat{p}_2(1-\hat{p}_2) \geq 10$$

(1) Determine the null and alternative hypotheses. They can be:

(i) Two tailed: $H_0 : p_1 = p_2$, $H_1 : p_1 \neq p_2$

(ii) Left-tailed: $H_0 : p_1 = p_2$, $H_1 : p_1 < p_2$

(iii) Right-tailed: $H_0 : p_1 = p_2$, $H_1 : p_1 > p_2$

(2) Select the level of significance $\alpha$

(3) Compute the test statistic:

$$z_0 = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

where $\hat{p} = \dfrac{x_1 + x_2}{n_1 + n_2}$

(4) Determine the critical value(s) and draw a conclusion:

(i) Two tailed:
The critical values are $-z_{\frac{\alpha}{2}}$ and $z_{\frac{\alpha}{2}}$, and we reject $H_0$ if $z_0 < -z_{\frac{\alpha}{2}}$ or if $z_0 > z_{\frac{\alpha}{2}}$

(ii) Left-tailed:
The critical value is $-z_\alpha$ and we reject $H_0$ if $z_0 < -z_\alpha$.

(iii) Right-tailed:
The critical value is $z_\alpha$ and we reject $H_0$ if $z_0 > z_\alpha$.

Let's do an example.

**Example 11.2.** *In clinical trials of Nasonex, 3774 adult and adolescent allergy patients (patients 12 years and older) were randomly divided into two groups. The patients in group 1 (experimental group) received 200 micrograms of Nasonex, while the patients in group 2 (control group) received a placebo. Of the 2103 patients in the experimental group, 547 reported headaches as a side effect. Of the 1671 patients in the control group, 368 reported headaches as a side effect.*

*Is there significant evidence to conclude that the proportion of Nasonex users who experienced headaches as a side effect is greater than the proportion in the control group at the a = 0.05 level of significance?*

**Solution.** Before starting the steps, let's extract the information about each group. For group 1 (experimental group) we have:

$$n_1 = 2103, \qquad x_1 = 547, \qquad \hat{p}_1 = \frac{x_1}{n_1} = \frac{547}{2103} = 0.2601$$

For group 2 (control group), we have

$$n_2 = 1671, \qquad x_2 = 368, \qquad \hat{p}_2 = \frac{x_2}{n_2} = \frac{368}{1671} = 0.2202$$

Now we go through the steps. We have:

(0) We verify the three conditions:

- The sampling method is independent because both groups have different individuals.
- For both samples, the population corresponds to adult and adolescents in the world. Then, $n_1$ and $n_2$ are well below the population size.
- We have

$$n_1\hat{p}_1(1-\hat{p}_1) = 2103 \cdot 0.2601 \cdot 0.7399 \approx 405 \geq 30$$
$$n_2\hat{p}_2(1-\hat{p}_2) = 1671 \cdot 0.2202 \cdot 0.7798 \approx 287 \geq 30$$

Hence, all the conditions are satisfied.

(1) We want to learn if group 1 has a higher proportion than group 2. Then, the null and alternative hypotheses are:

$$H_0 : p_1 = p_2 \qquad H_1 : p_1 > p_2$$

Then, the hypothesis test is right-tailed.

(2) The level of significance is given: $\alpha = 0.05$

(3) To compute the test statistic we need to compute:

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{547 + 368}{2103 + 1671} = 0.2424$$

Then, the test statistic is:

$$\begin{aligned} z_0 &= \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \\ &= \frac{0.2601 - 0.2202}{\sqrt{0.2424 \cdot 0.7576}\sqrt{\frac{1}{2103} + \frac{1}{1671}}} \\ &= 2.85 \end{aligned}$$

(4) The critical value can be obtained from the last row of the $t$-table. We obtain:

$$z_\alpha = z_{0.05} = 1.645$$

Then, we have

$$z_0 = 2.85 > z_\alpha = 1.645$$

Hence, we reject the null hypothesis.

□

## 11.2 Test hypotheses regarding the difference of two independent means

Similarly to the test for population proportions, we base this test on the hypothesis tests for the population mean with one sample. We obtain the following steps.

(0) Verify conditions:

- The samples are independent
- The samples are drawn from normally distributed populations, or each sample size is greater than 30 (that is, $n_1 \geq 30$ and $n_2 \geq 30$)
- For each sample, the sample size is less than 5% of the population size.

(1) Determine the null and alternative hypotheses. They can be:

(i) Two-tailed: $H_0 : \mu_1 = \mu_2$, $H_1 : \mu_1 \neq \mu_2$

(ii) Left-tailed: $H_0 : \mu_1 = \mu_2$, $H_1 : \mu_1 < \mu_2$

(iii) Right=tailed: $H_0 : \mu_1 = \mu_2$, $H_1 : \mu_1 > \mu_2$

(2) Select the level of significance $\alpha$

(3) Compute the test statistic:

$$t_0 = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

(4) Determine the critical value(s) and draw a conclusion:

(i) Two tailed:
The critical values are $-t_{\frac{\alpha}{2}}$ and $t_{\frac{\alpha}{2}}$, where $t_{\frac{\alpha}{2}}$ is computed using the smaller of $n_1 - 1$ and $n_2 - 1$ degrees of freedom. We reject $H_0$ if $t_0 < -t_{\frac{\alpha}{2}}$ or if $z_0 > t_{\frac{\alpha}{2}}$.

(ii) Left-tailed:
The critical value is $-t_\alpha$, where $t_\alpha$ is computed using the smaller of $n_1 - 1$ and $n_2 - 1$ degrees of freedom. We reject $H_0$ if $t_0 < -t_\alpha$

(iii) Right-tailed:
The critical value is $t_\alpha$, where $t_\alpha$ is computed using the smaller of $n_1 - 1$ and $n_2 - 1$ degrees of freedom. We reject $H_0$ if $z_0 > z_\alpha$

We end this chapter with an example.

**Example 11.3.** *In the Spacelab Life Sciences 2 payload, 14 male rats were sent to space. Upon their return, the red blood cell mass (in milliliters) of the rats was determined. A control group of 14 male rats was held under the same conditions (except for space flight) as the space rats, and their red blood cell mass was also determined when the space rats returned.*

*The mean and standard deviation of the red blood cell mass of the rats sent to space are* $\overline{x}_1 = 7.881$ *and* $s_1 = 1.017$*, respectively; and for the control group, the mean and standard deviation are* $\overline{x}_2 = 8.430$ *and* $s_2 = 1.005$*. You can assume that the red blood cell mass follows a normal distribution for both groups of rats.*

*Does the evidence suggest that the flight animals have a different red blood cell mass from the control animals at the* $\alpha = 0.05$ *level of significance?*

**Solution.** We follow the steps detailed above.

(0) We verify the conditions:

- The samples are independent because they involve different groups of rats
- The samples are drawn from normally distributed populations
- For each sample, the sample size is $n_1 = n_2 = 14$ and this number is considerably smaller than the total number of rats.

(1) Since we want to learn if the samples are different, we use a two-tailed test. Hence,

$$H_0 : \mu_1 = \mu_2, \qquad H_1 : \mu_1 \neq \mu_2$$

(2) The level of significance is $\alpha = 0.05$

(3) We compute the test statistic, observe:

$$\overline{x}_1 = 7.881, \qquad s_1 = 1.017, \qquad n_1 = 14$$
$$\overline{x}_2 = 8.430, \qquad s_2 = 1.005, \qquad n_2 = 14$$

Then, we obtain:

$$t_0 = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{7.881 - 8.430}{\sqrt{\frac{1.017^2}{14} + \frac{1.005^2}{14}}} = -1.437$$

(4) The critical values are computed with $n_1 - 1 = n_2 - 1 = 13$ degrees of freedom. We have

$$t_{\frac{\alpha}{2}} = t_{0.025} = 2.160$$

and

$$t_0 = -1.437 > t_{0.025} = -2.160$$

Then, we do not reject $H_0$.

□