Minimizing Delay in Supermarket-Checkout Systems

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William and Mary
Joint work with Siva Theja Maguluri

Mathematics Colloquium, April 1st, 2022

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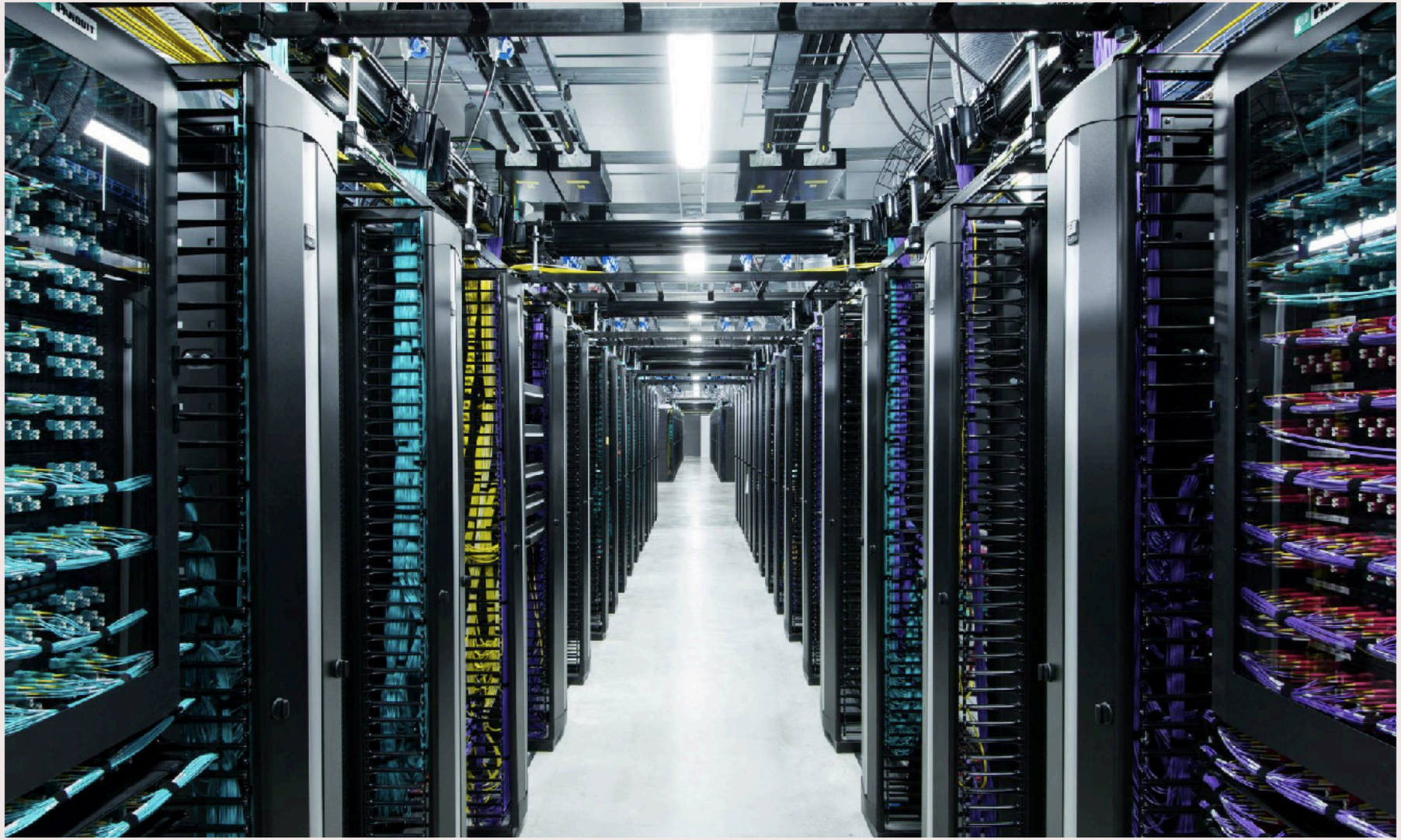
Motivation

Data Centers



Data Storage and Production

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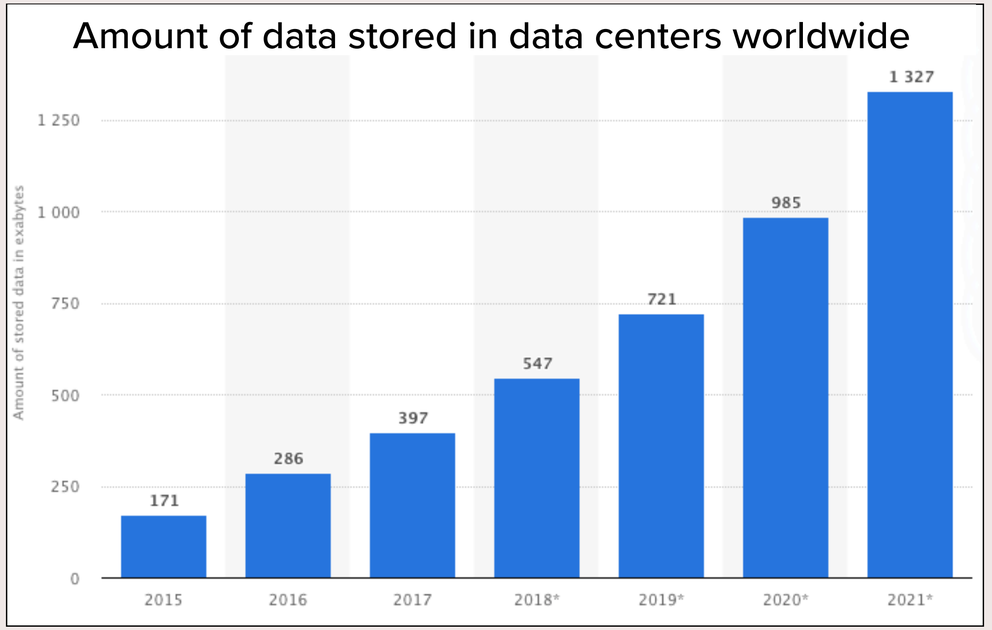

Source: https://www.statista.com/statistics/638613/worldwide-data-center-storage-

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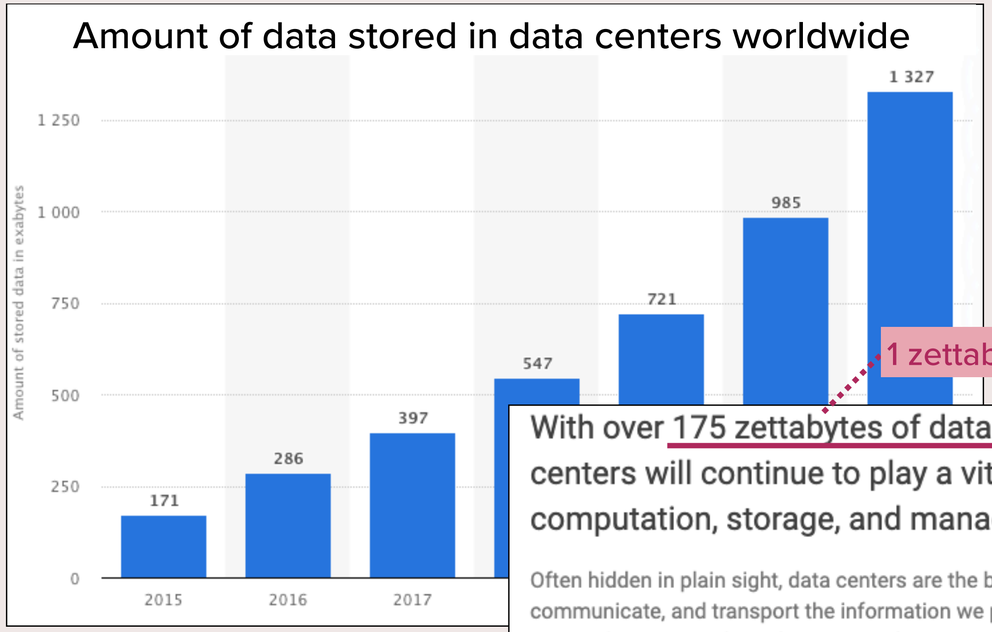

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1 zettabyte = 1B terabytes

With over 175 zettabytes of data expected by 2025, data centers will continue to play a vital role in the ingestion, computation, storage, and management of information.

Often hidden in plain sight, data centers are the backbone of our internet. They store, communicate, and transport the information we produce every single day. The more data we create, the more vital our data centers become.

But many of today's data centers are clunky, inefficient, and outdated. To keep them running, data center operators, from FAMGA to colocation providers, are working on upgrading them to fit our ever-changing world.

Source: The future of data center, CB insights, available at https://www.cbinsights.com/research/future-of-data-centers/

Data Storage and Production

Are There Data Center Storage Capacity Constraints?

There's no argument in the data world that there is, or at least will be, a need to continue data center and data storage expansion, but what that future looks like is unknown. And while currently the world's data centers store about 1,327 exabytes of data, and we're producing about 2.5 exabytes of data daily, we're not in danger of running out of data storage space just yet.

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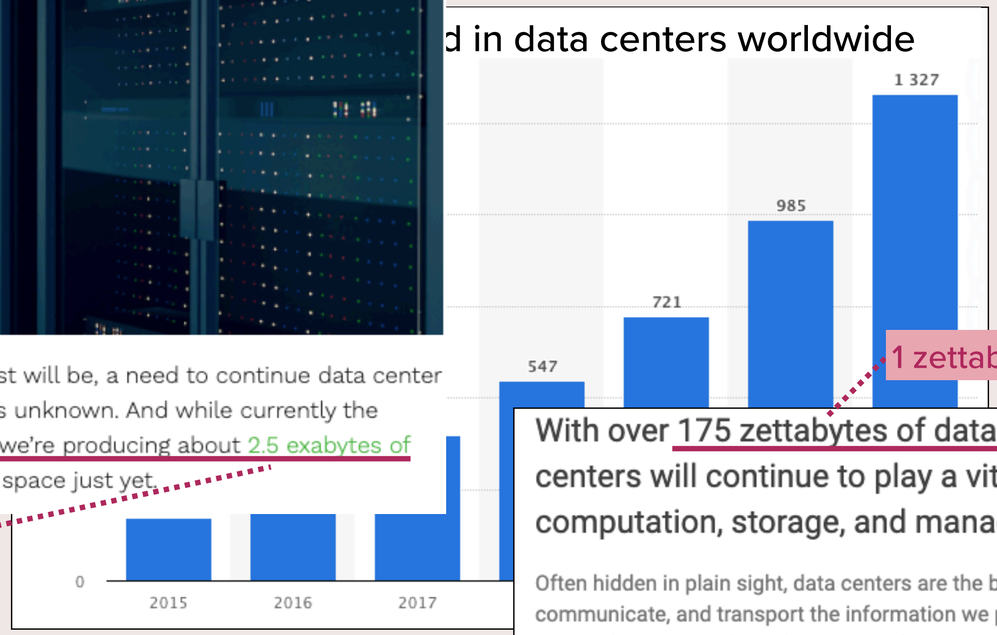

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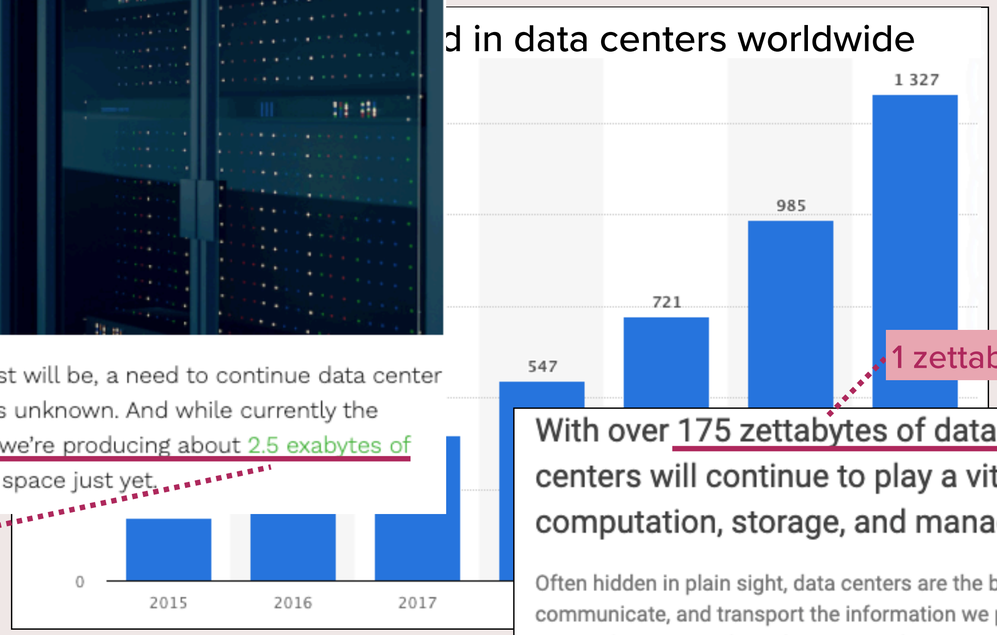

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29 terabytes per second

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- ↓0.59% # searches per user
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How? Queueing theory

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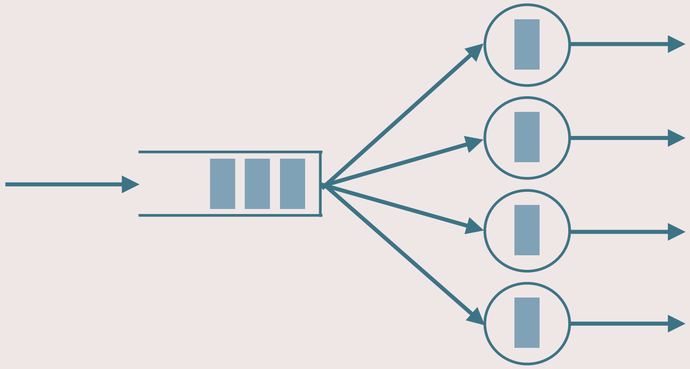

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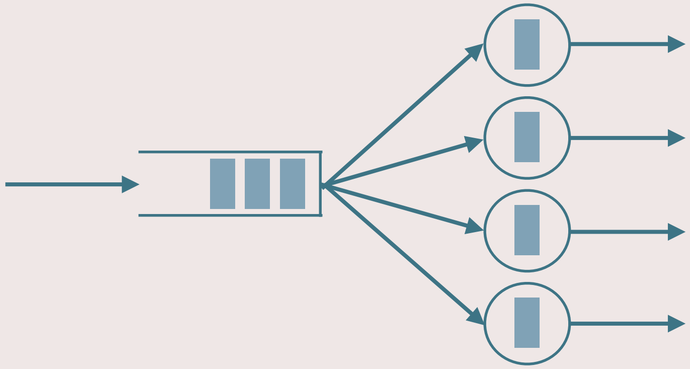

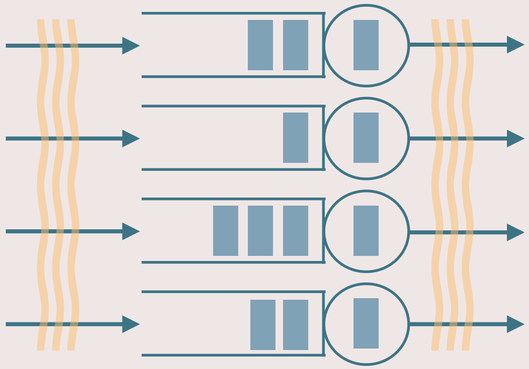

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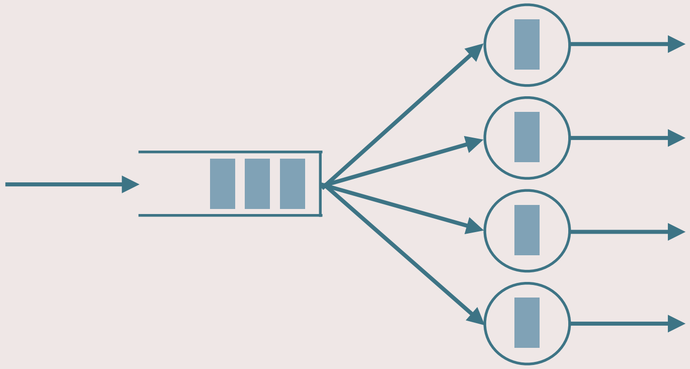

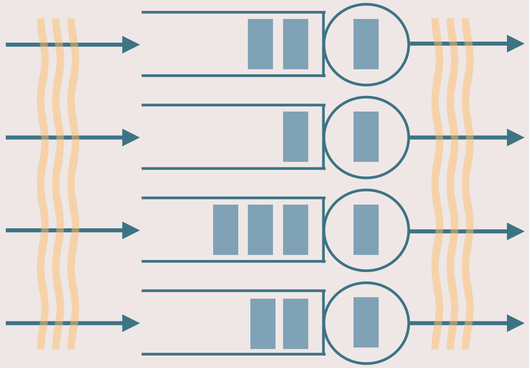

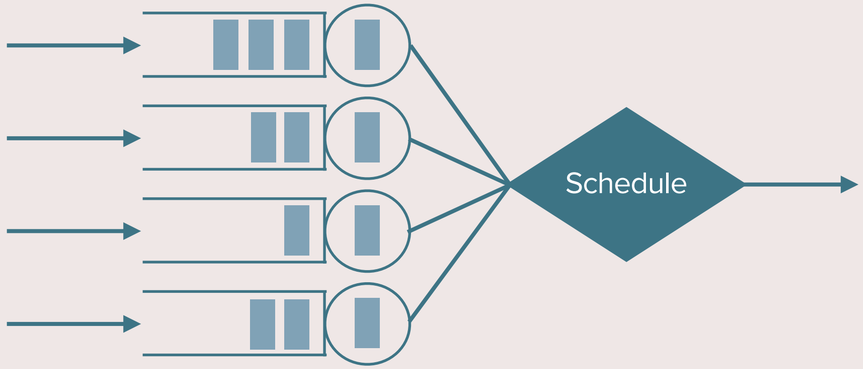

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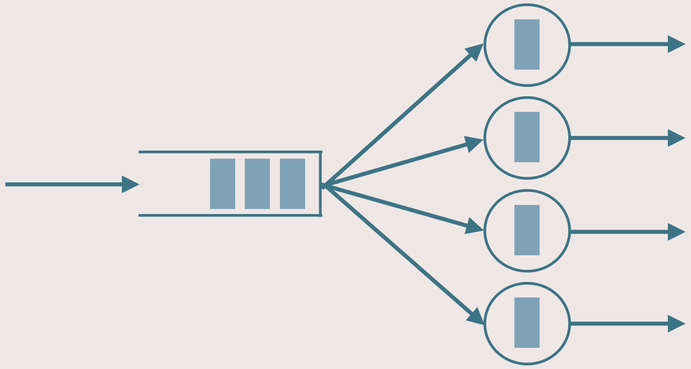

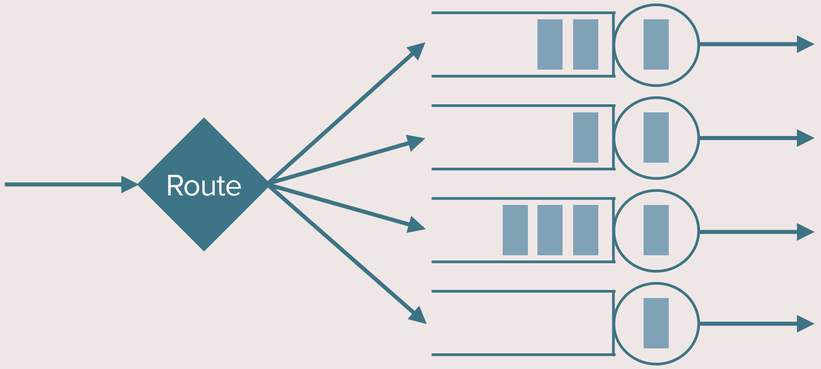

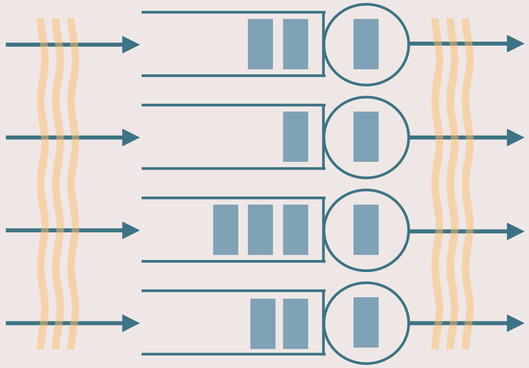

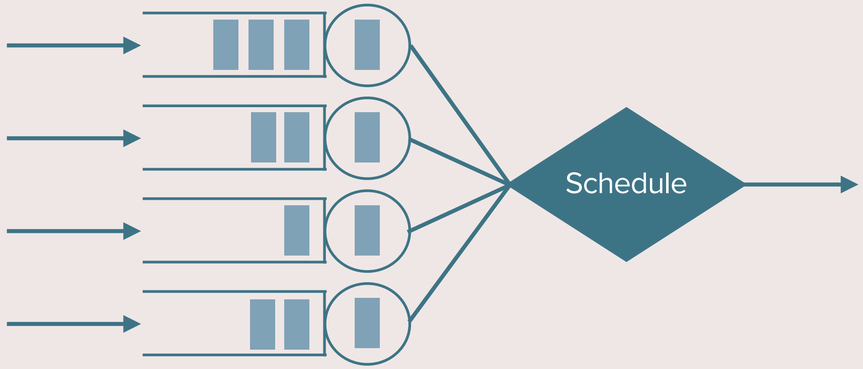

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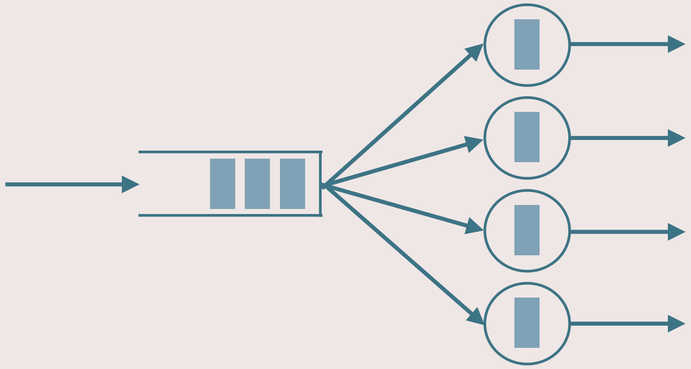

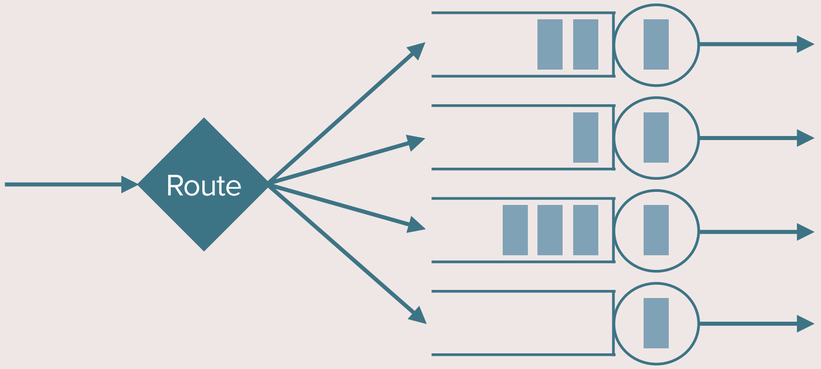

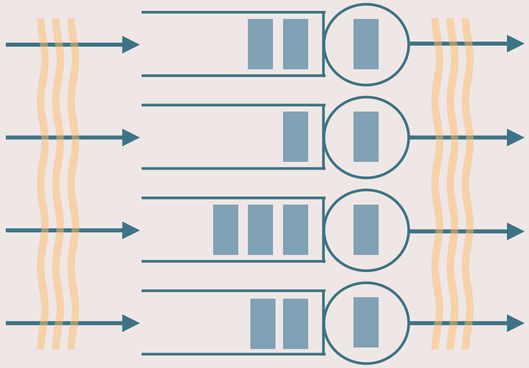

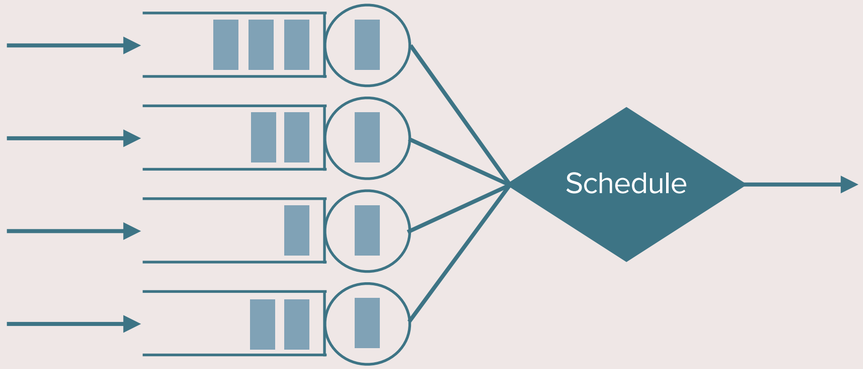

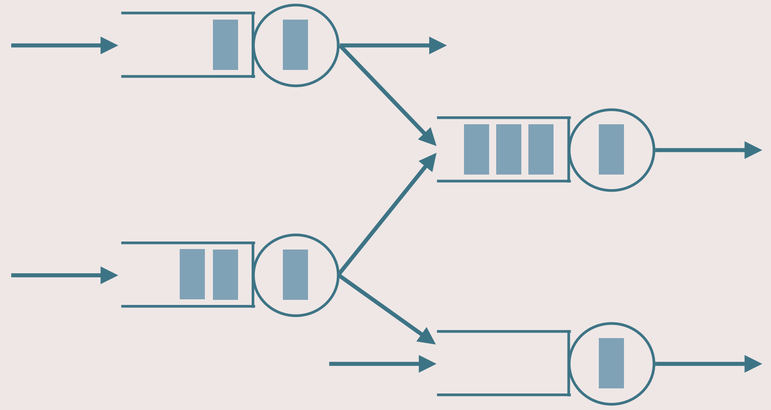

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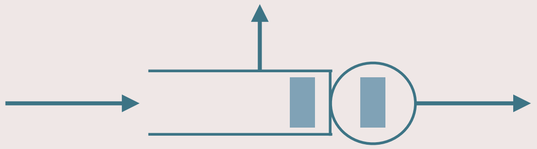

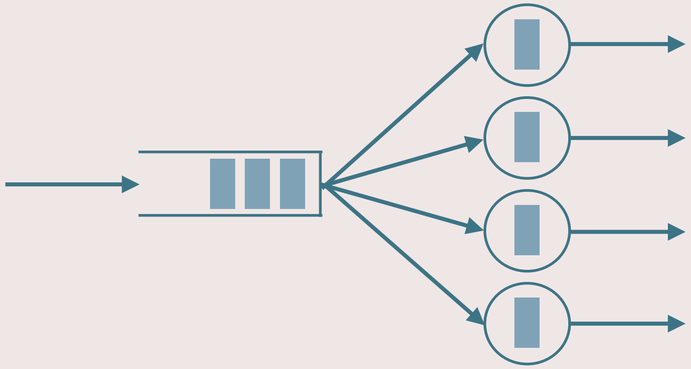

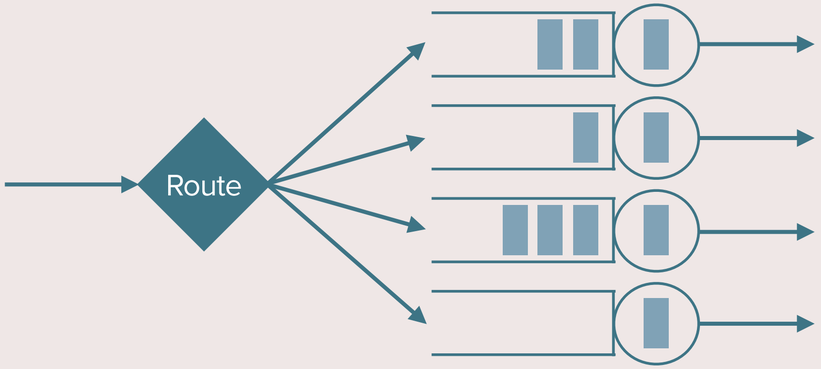

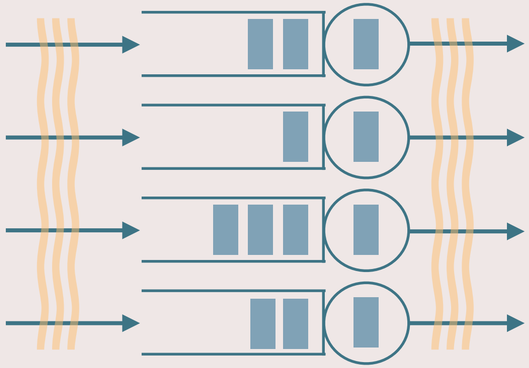

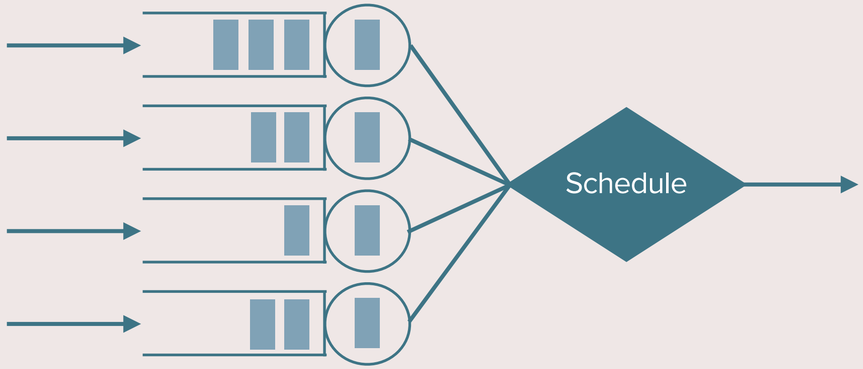

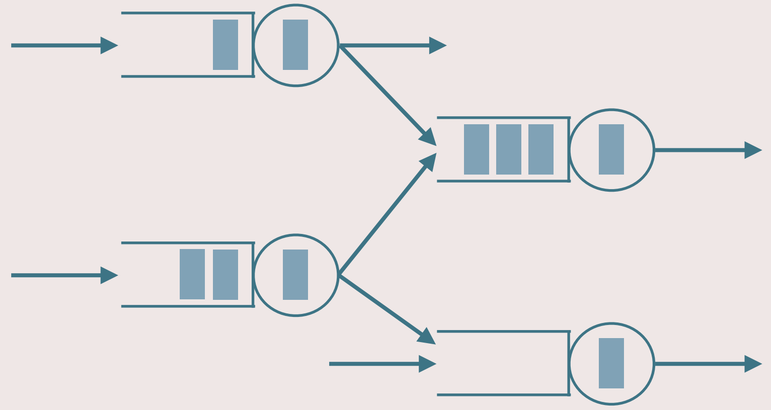

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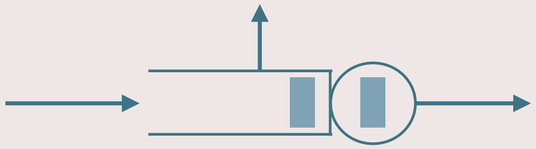

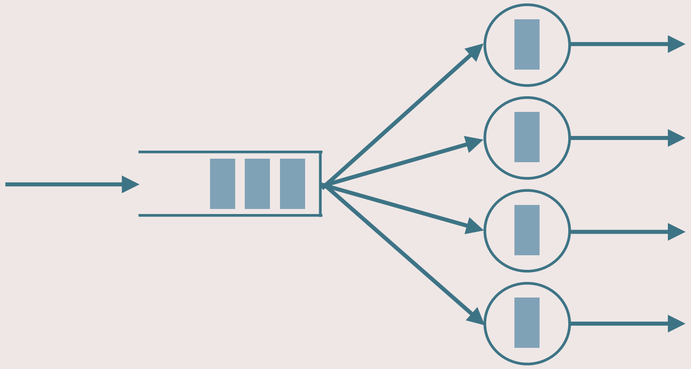

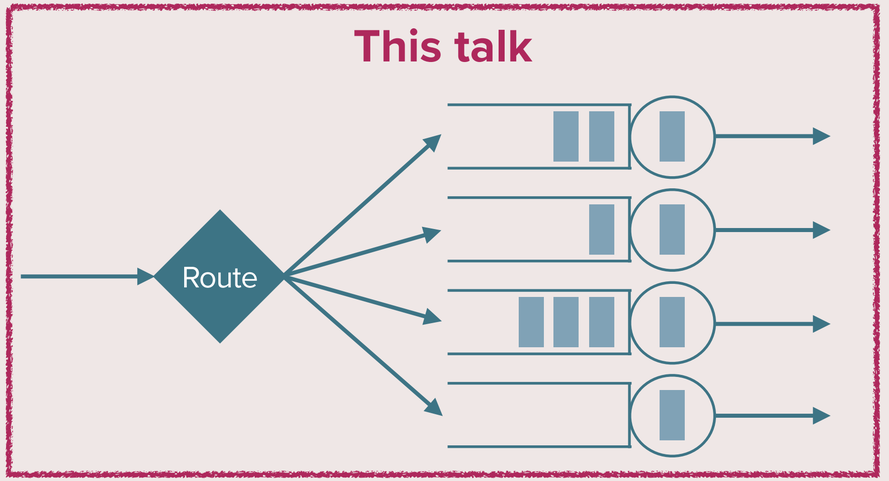

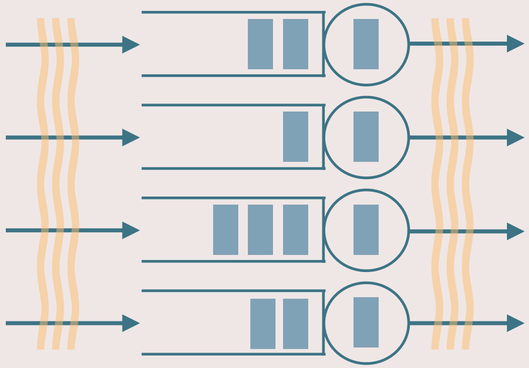

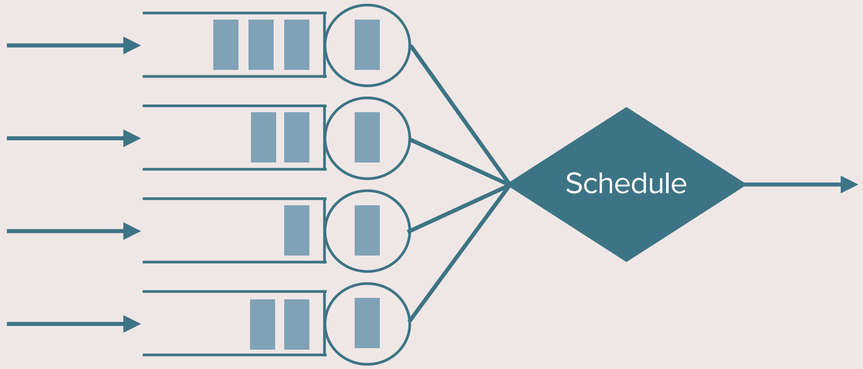

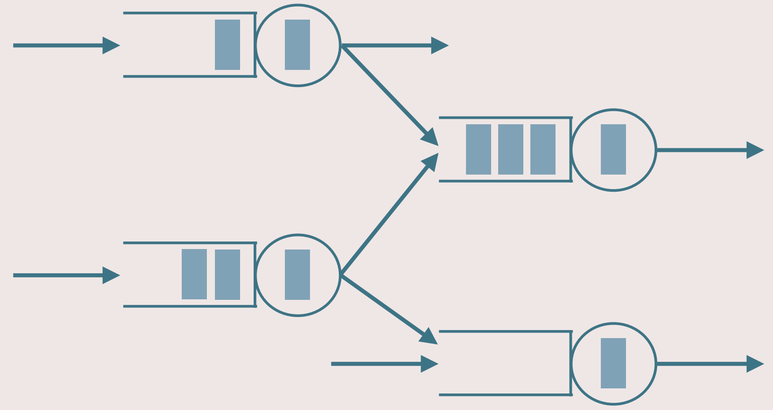

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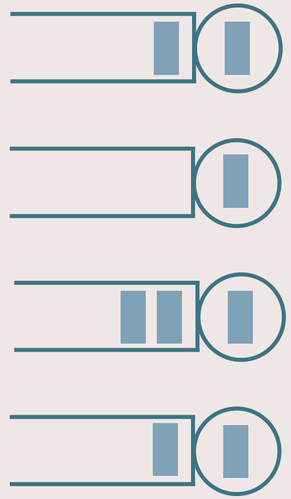
- n servers with an infinite buffer



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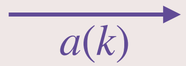

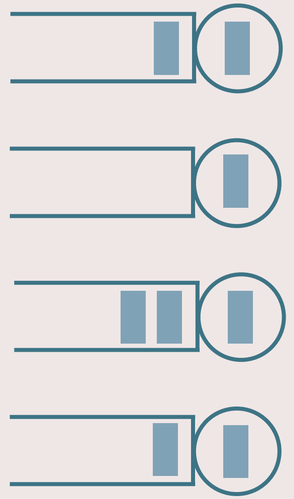

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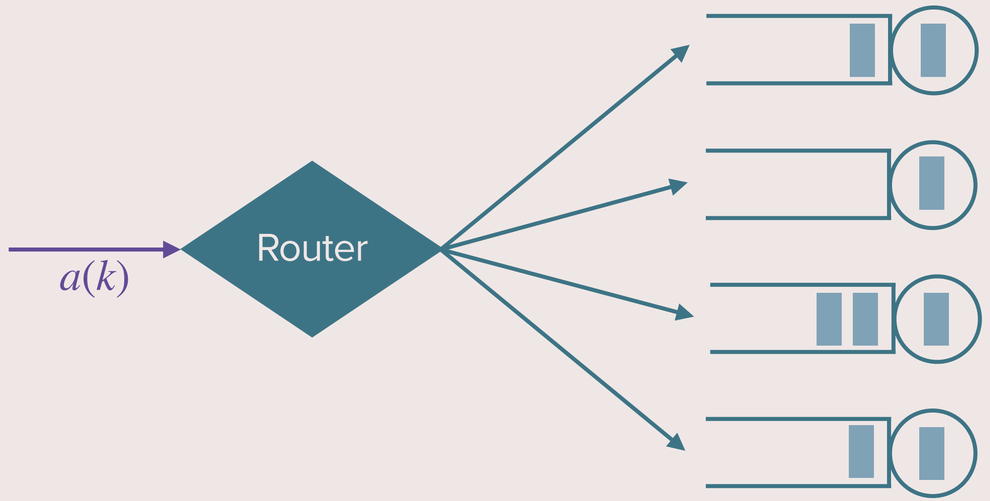

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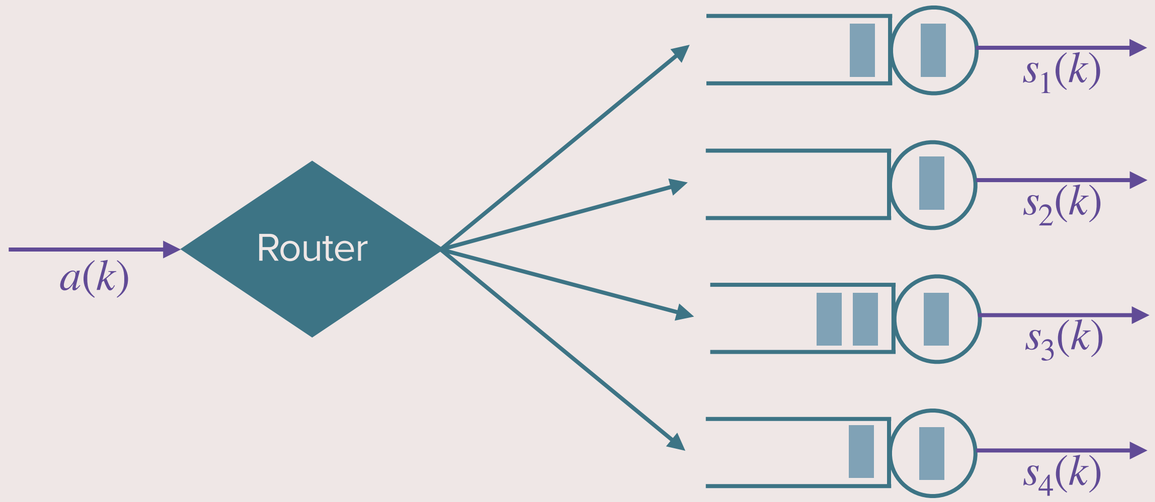
Service $s_1(k), s_2(k), \ldots, s_n(k)$:

- Potential service is a sequence of i.i.d. random variables
- Service processes to different queues are correlated



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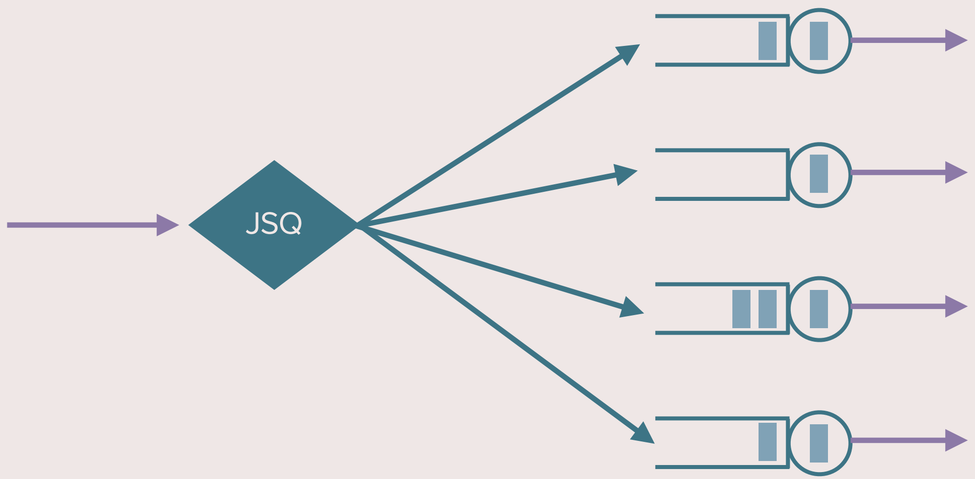

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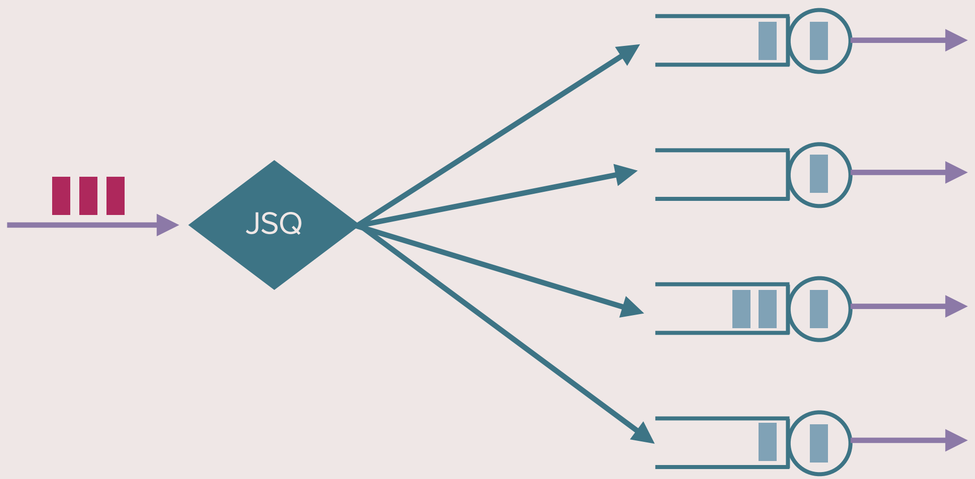

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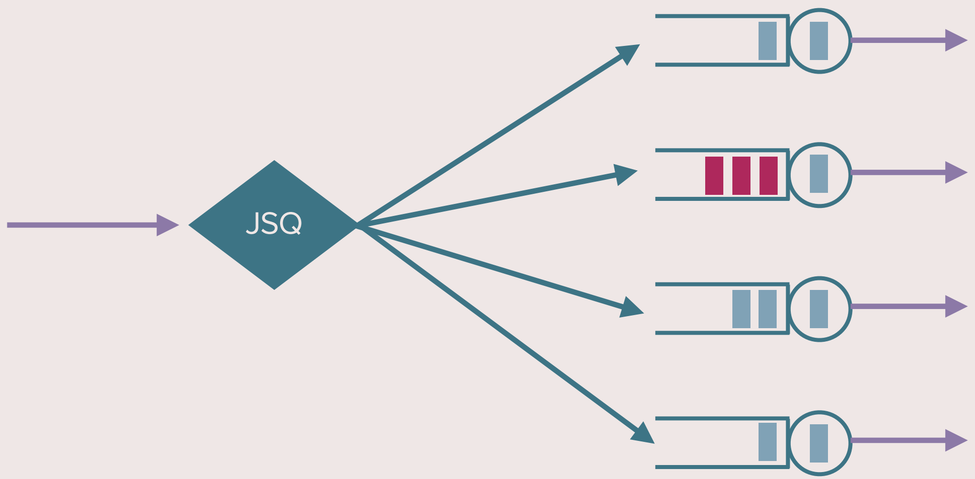

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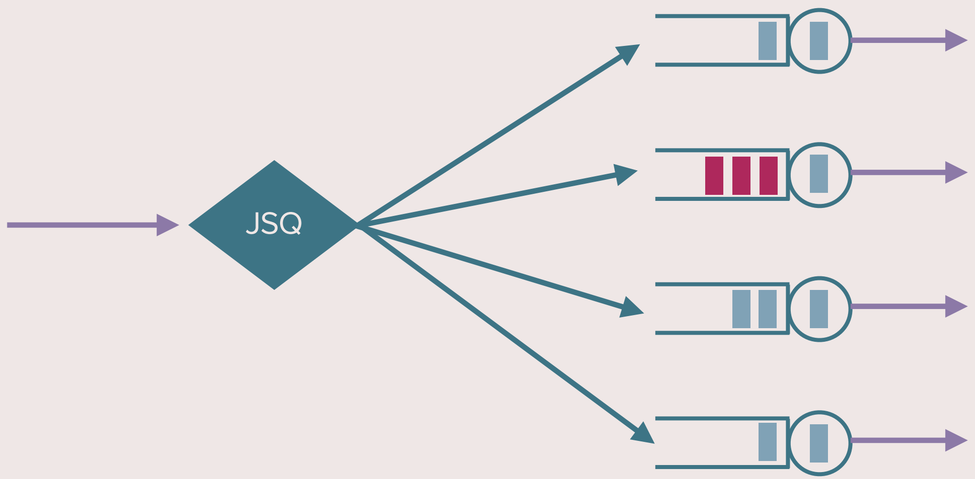

Power of d choices, or JSQ(d):
Given an integer $d \in [1,n] \cap \mathbb{Z}$, in each time slot:

(1) Select d servers uniformly at random

(2) Route arrivals to the shortest queue among those d

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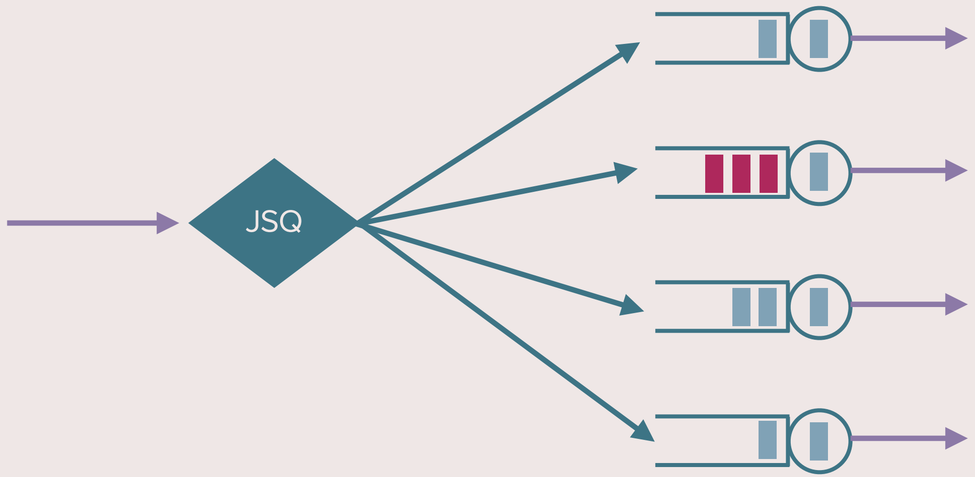

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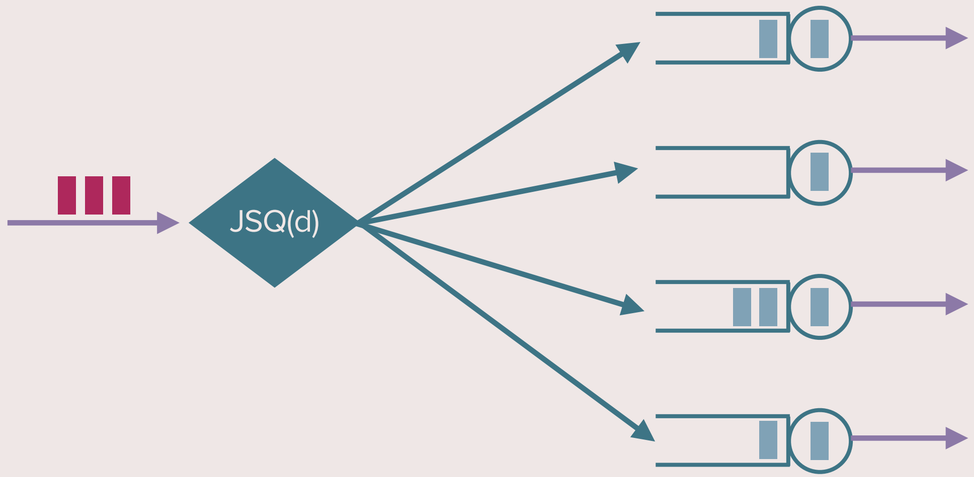

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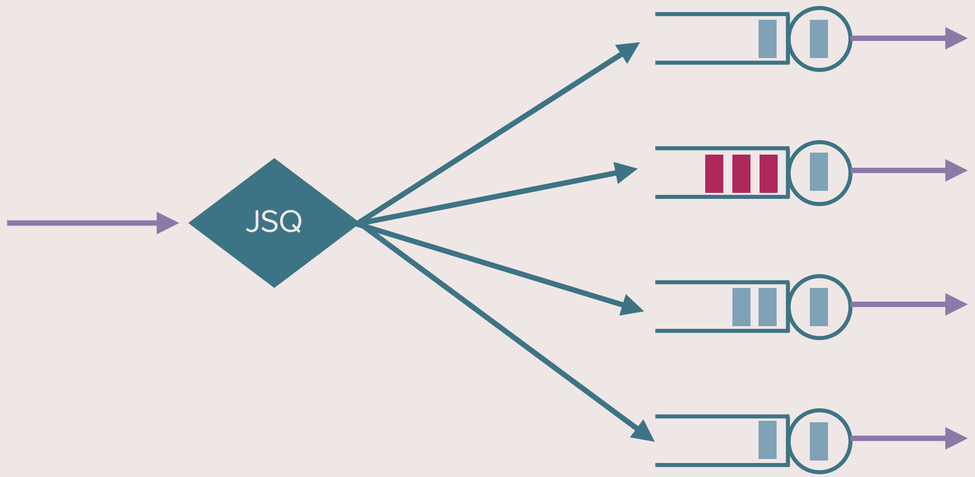

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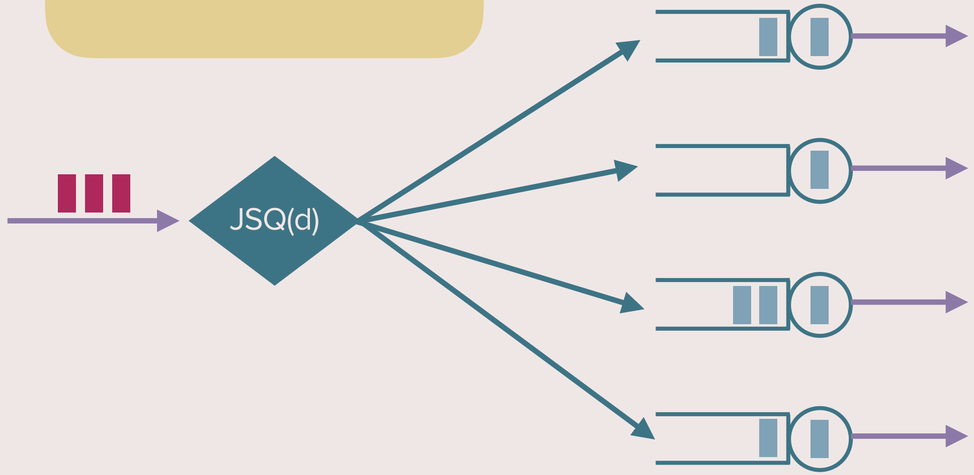

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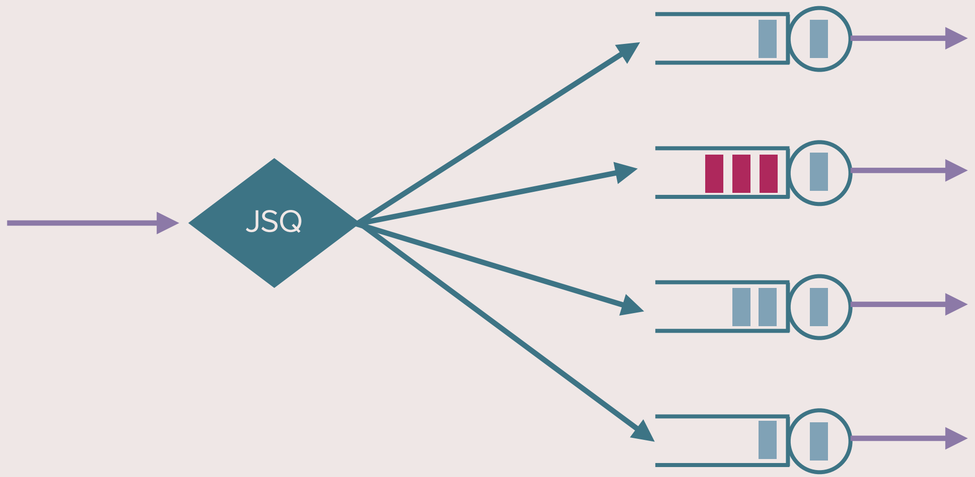

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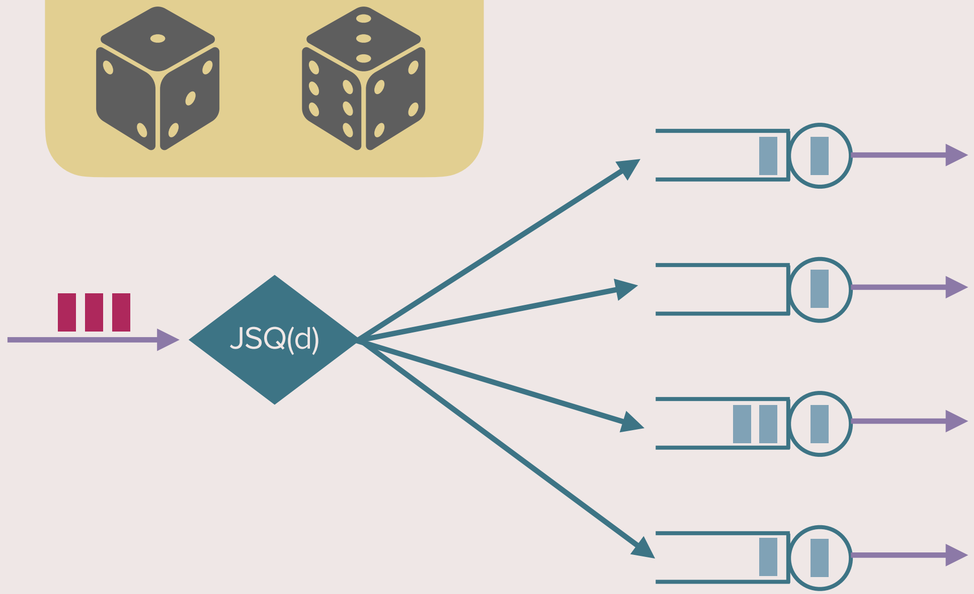

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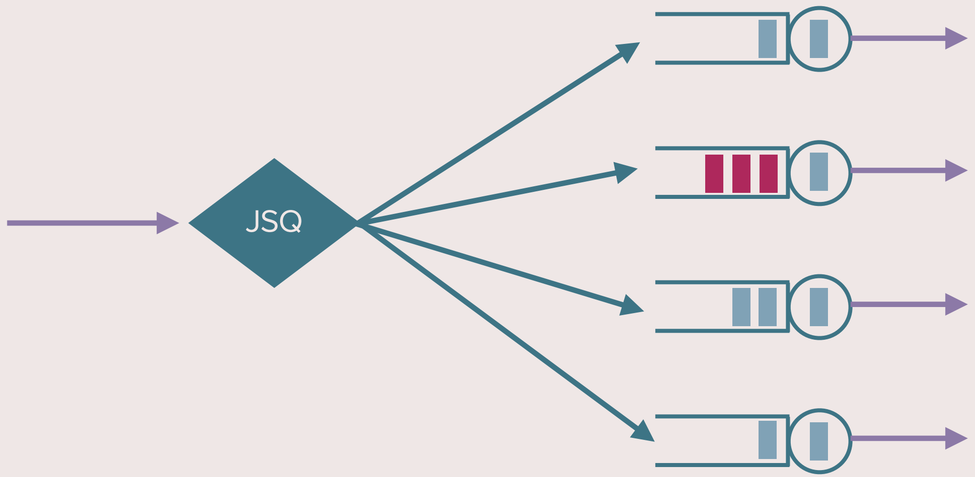

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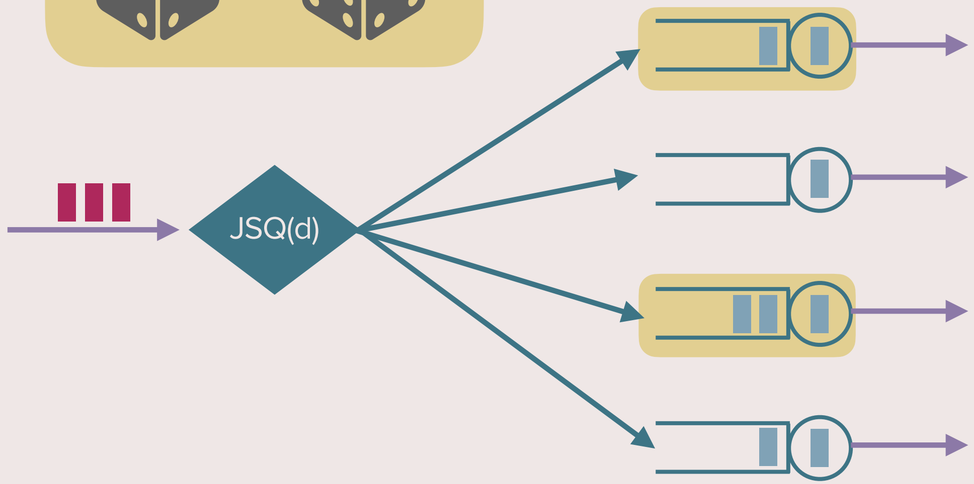

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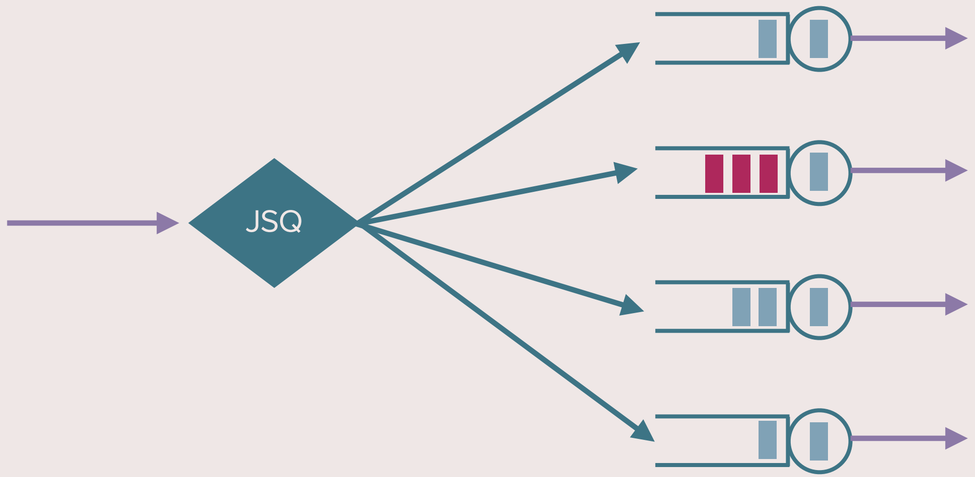

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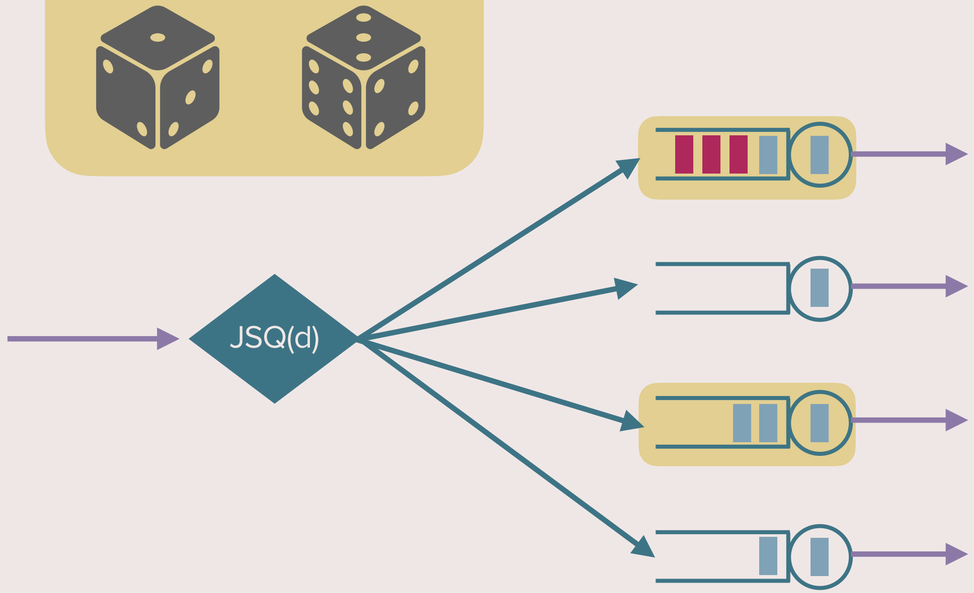

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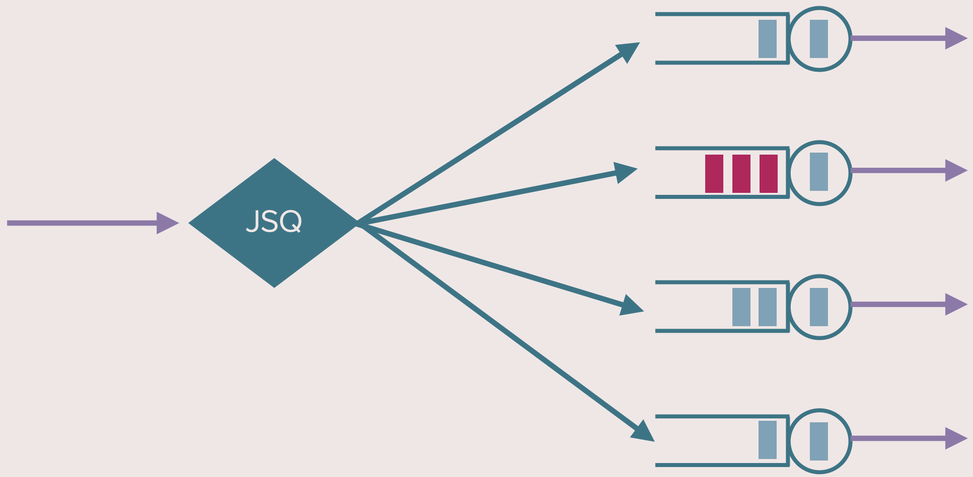

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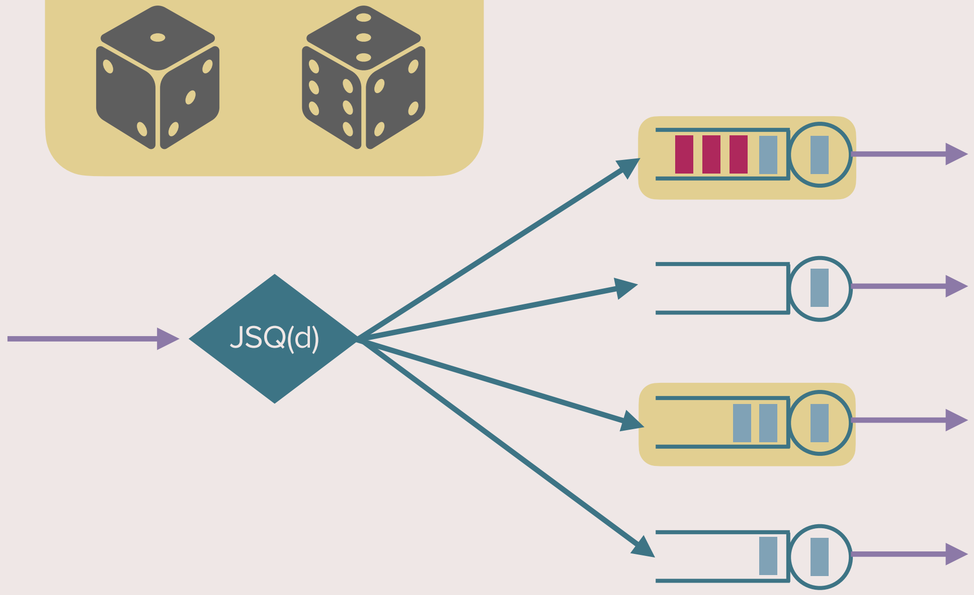

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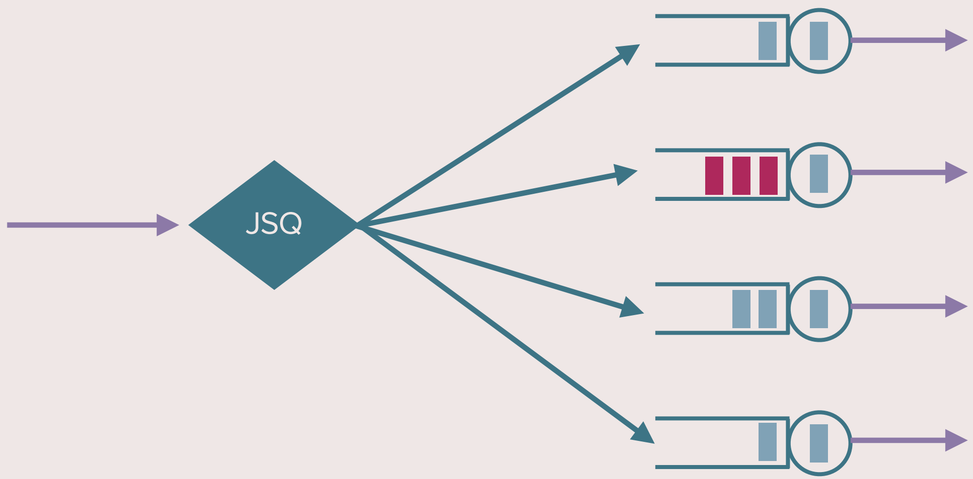

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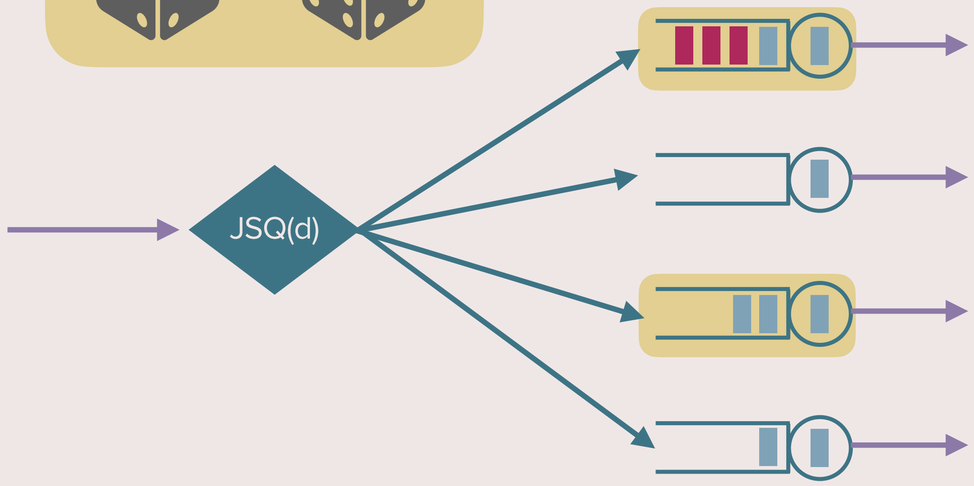

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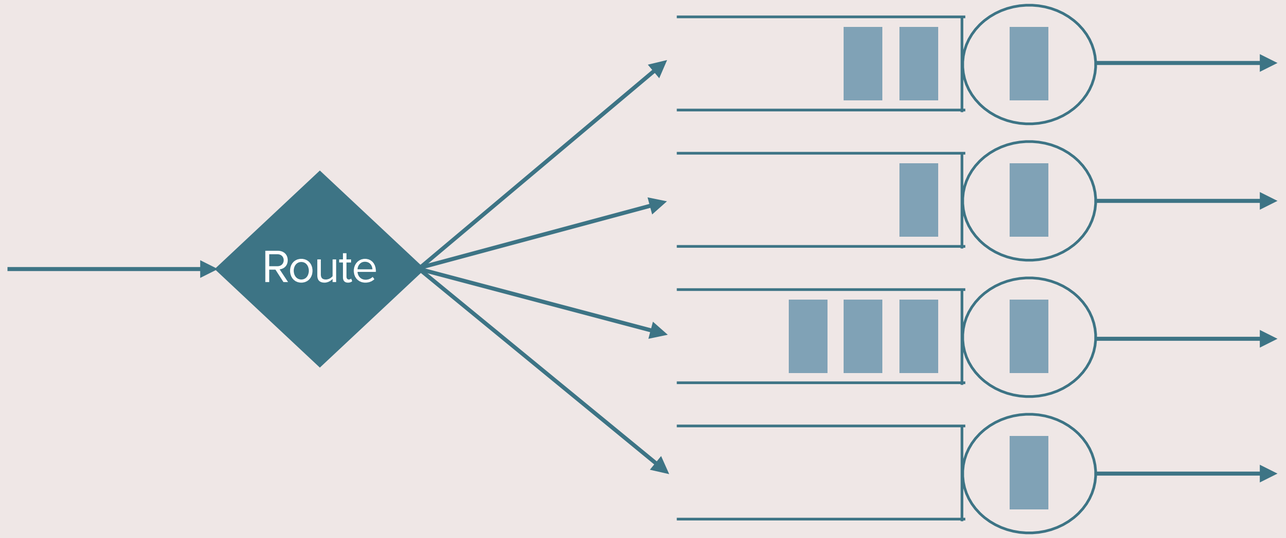

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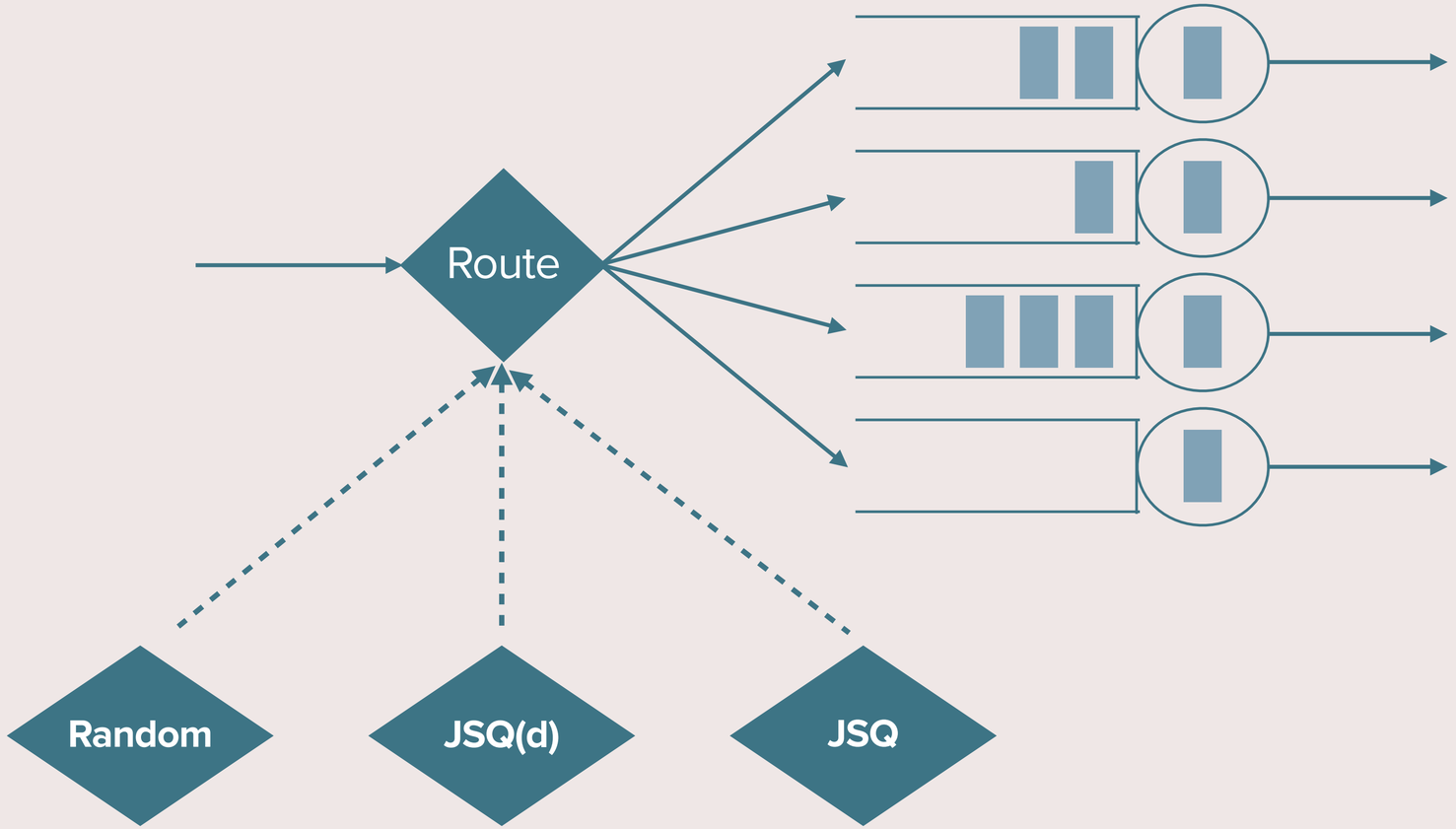

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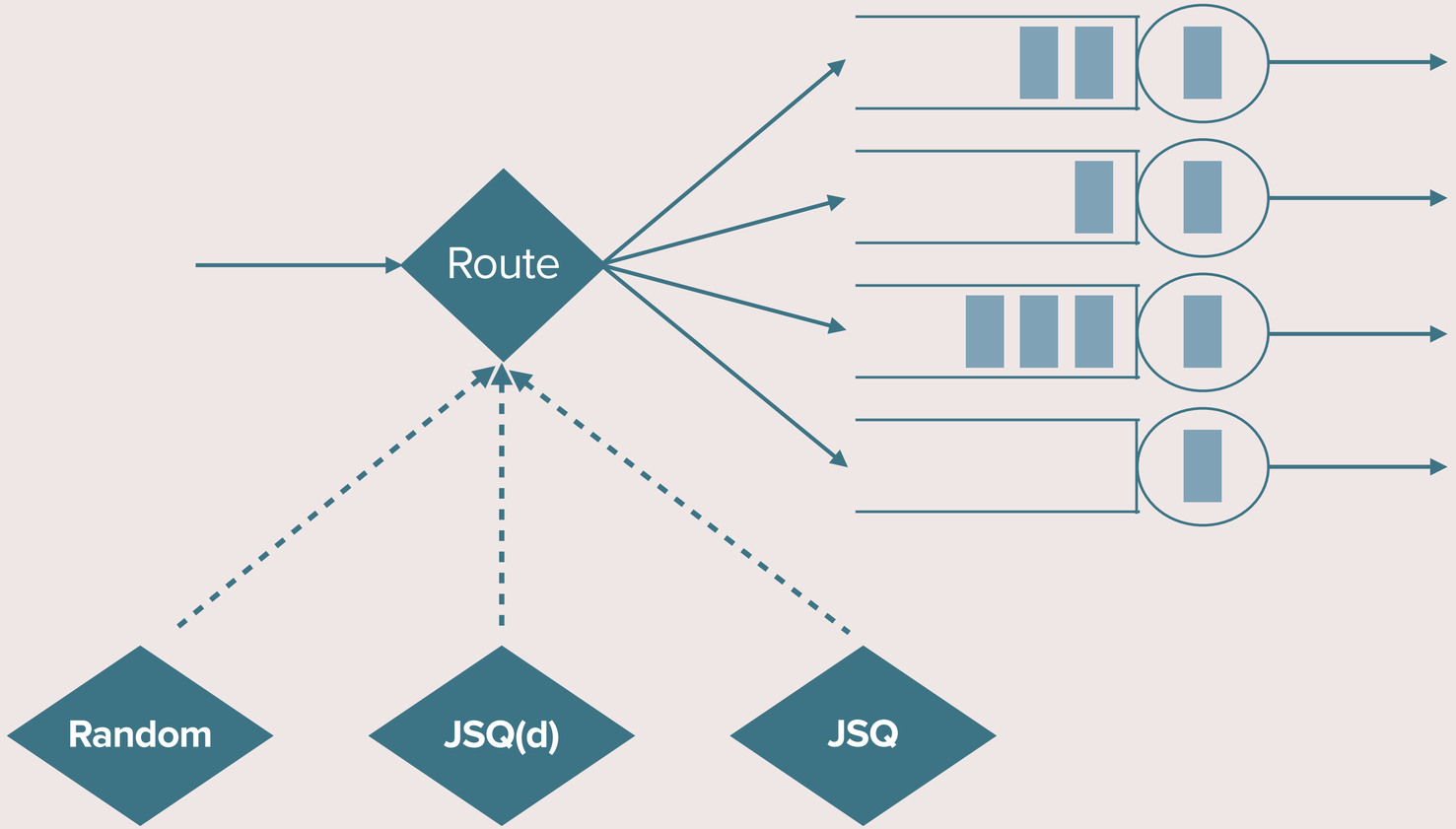

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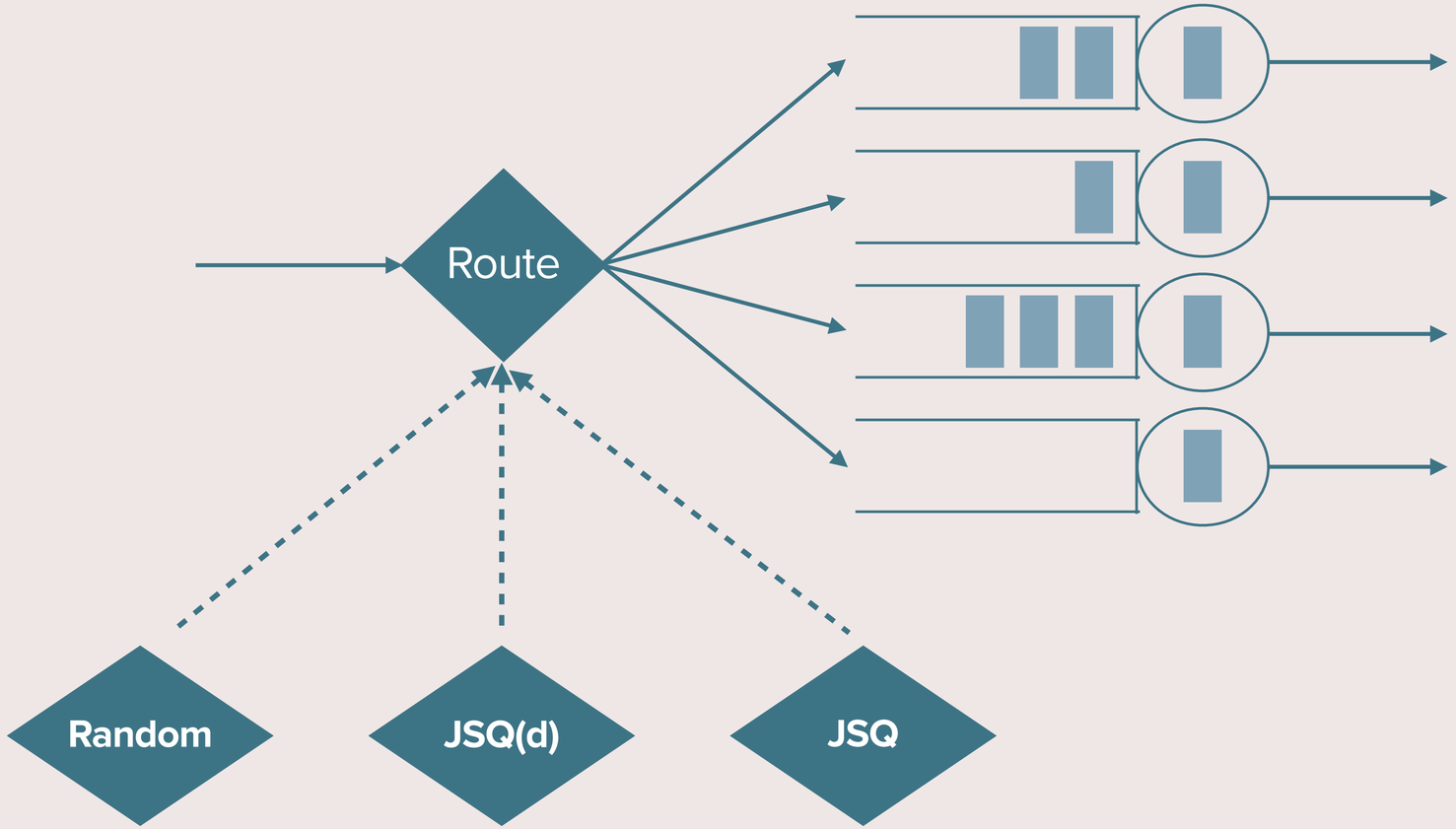

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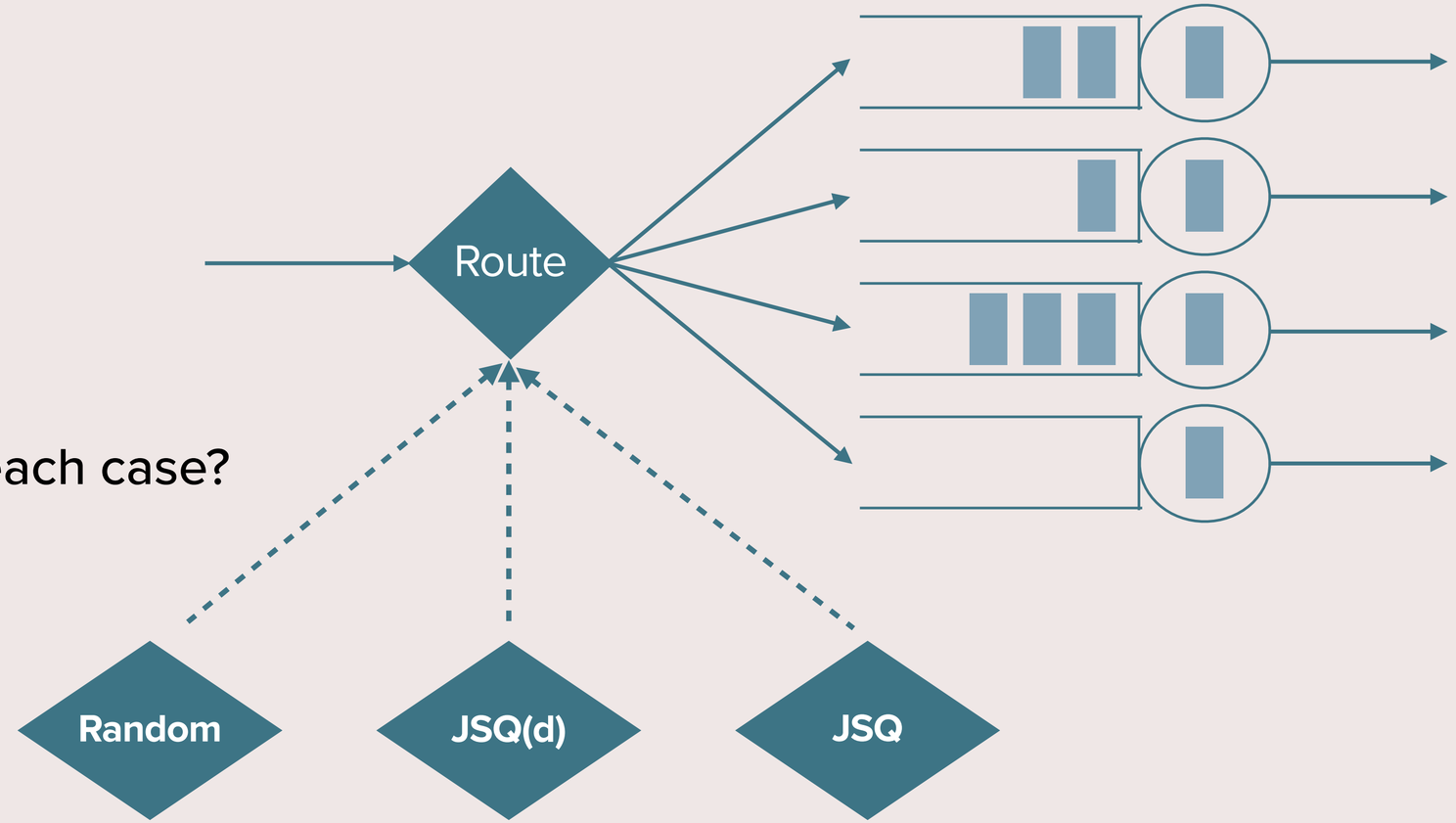

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Part 2: How to Solve?	• Stability • Heavy-Traffic: The CRP condition • Analysis of a single-server queue • The MGF method
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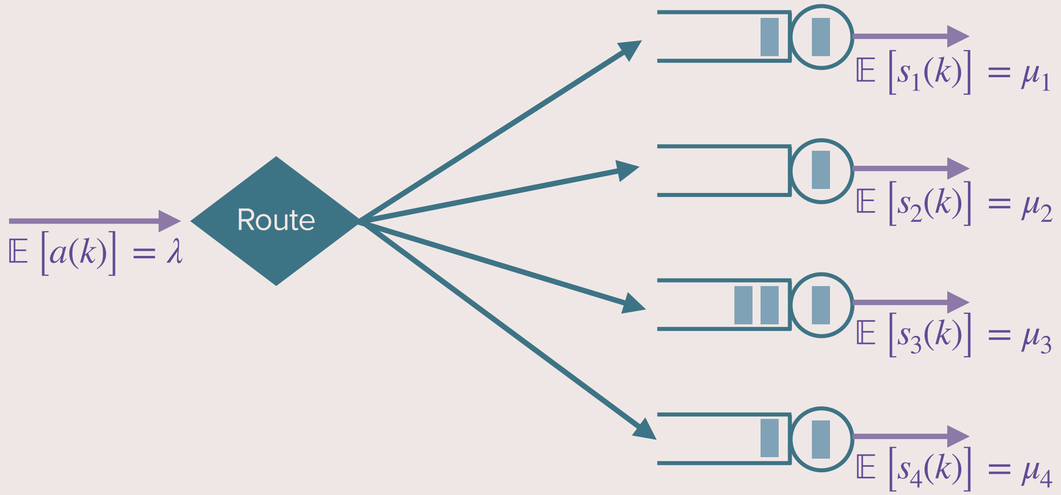

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Stability

Definition: Capacity region

Set of **arrival rates** so that there **exists** a routing algorithm such that the queue lengths are positive recurrent (stable), i.e., so that

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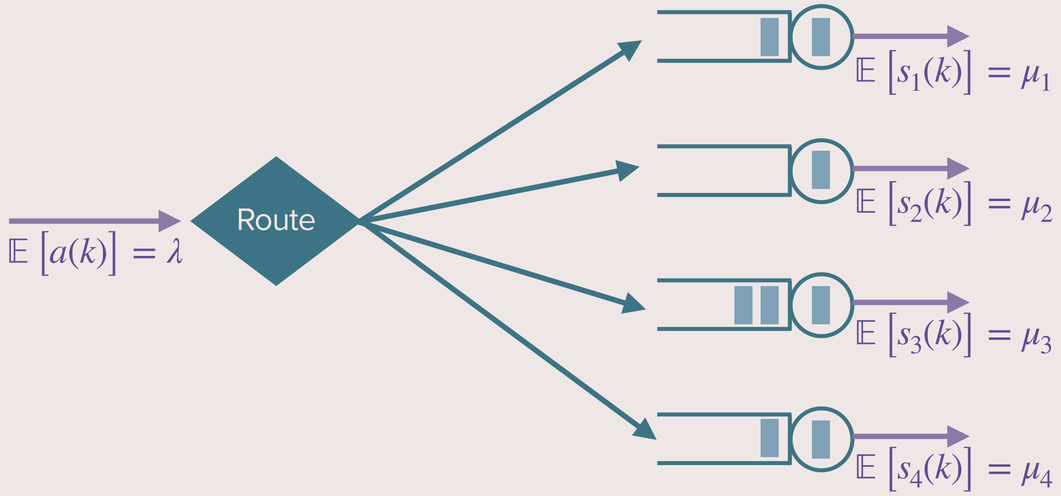

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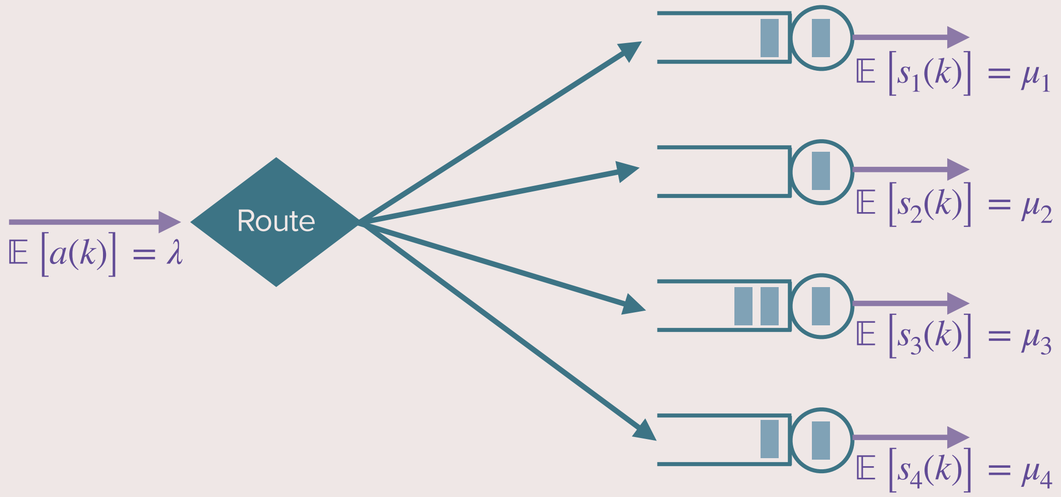

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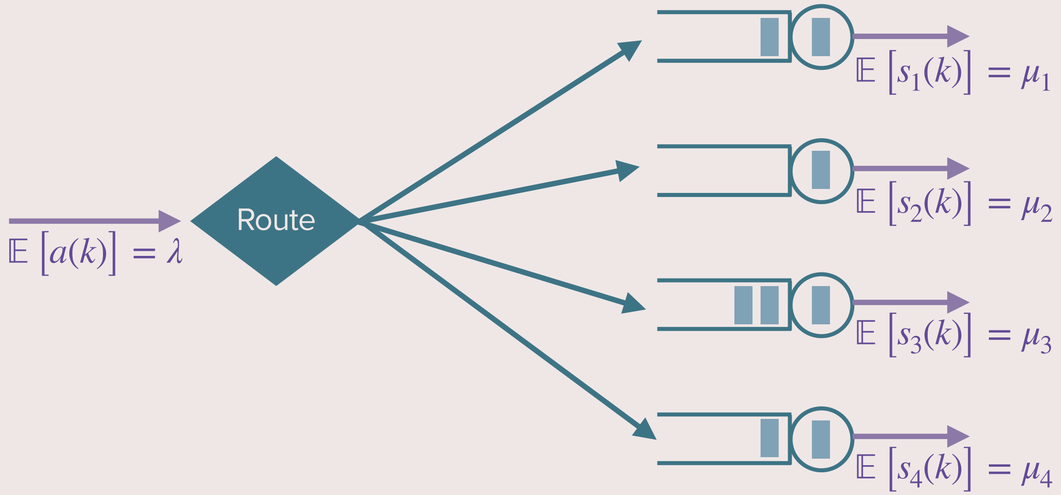

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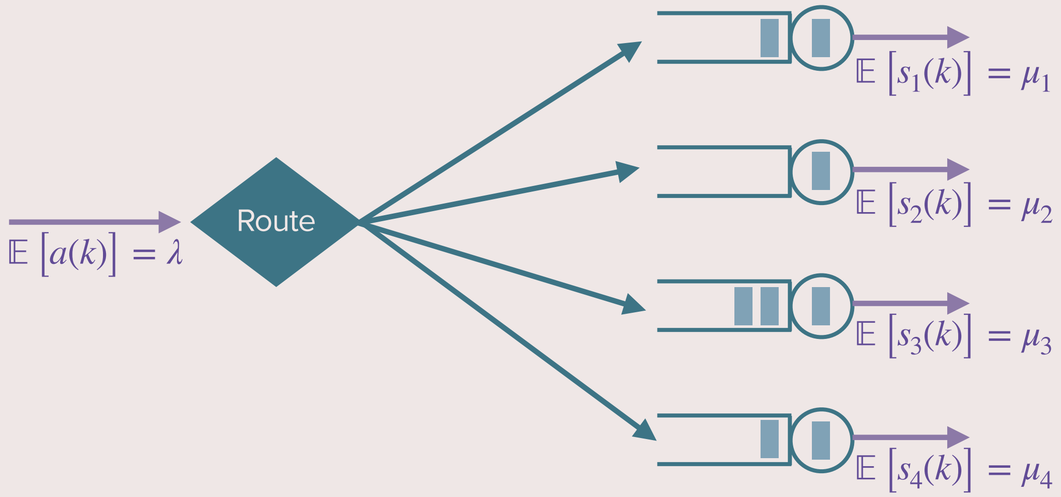

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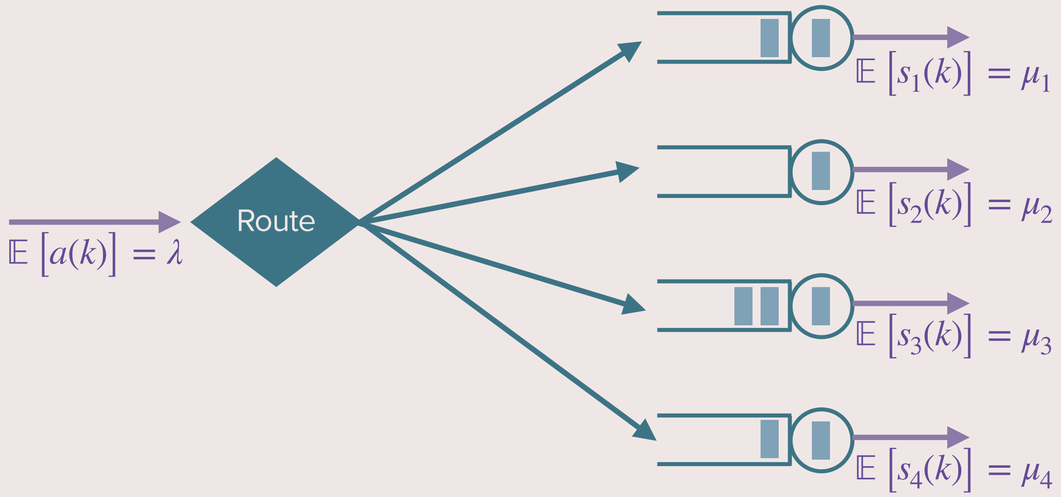

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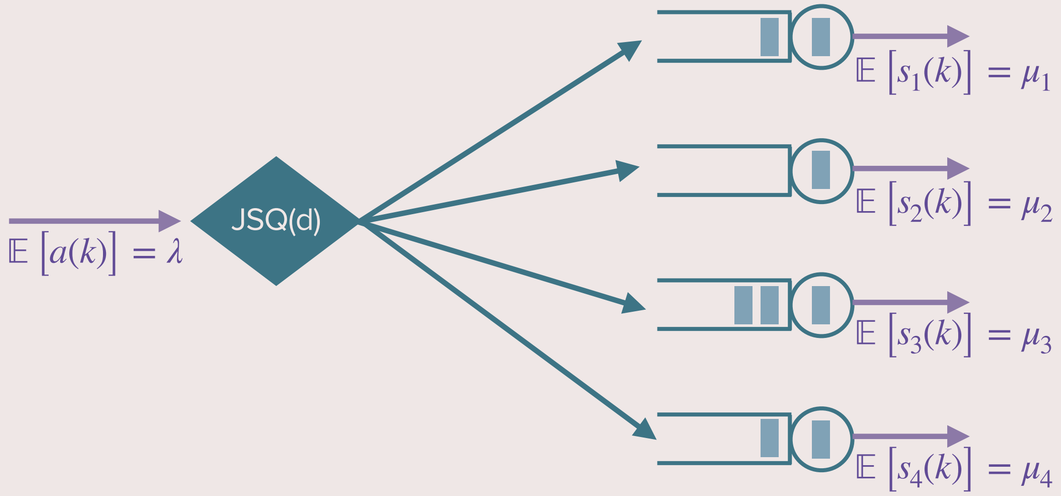
Should it be better than random?

JSQ(d) When Servers are Different

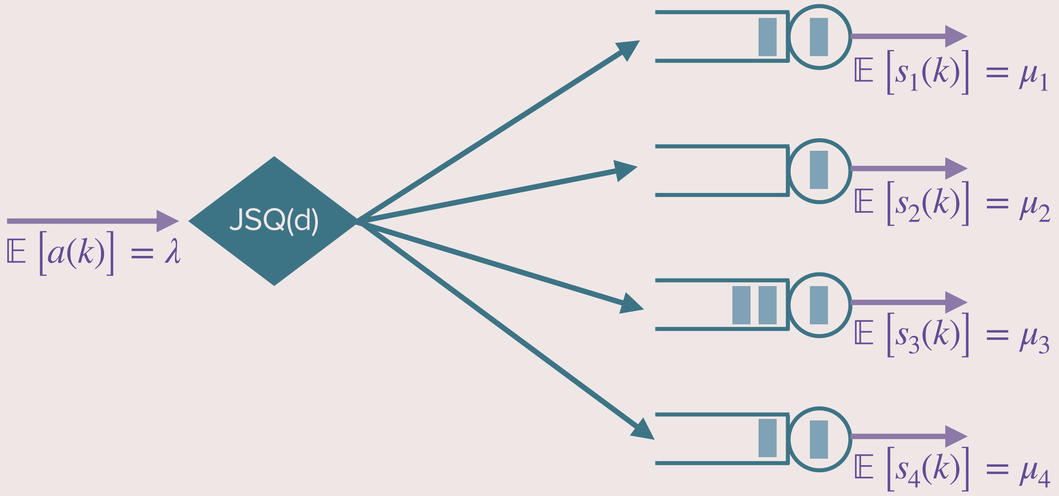


JSQ(d) When Servers are Different

Theorem: [HL, Maguluri 2020]

Let $1 \leq d \leq n-1$. Define $x_i = \dfrac{\mu_i}{\mu_\Sigma}$, and $y_i = \dfrac{\binom{i-1}{d-1}}{\binom{n}{d}}$.

Power-of-d is throughput optimal if and only if $\boldsymbol{x} \preccurlyeq \boldsymbol{y}$



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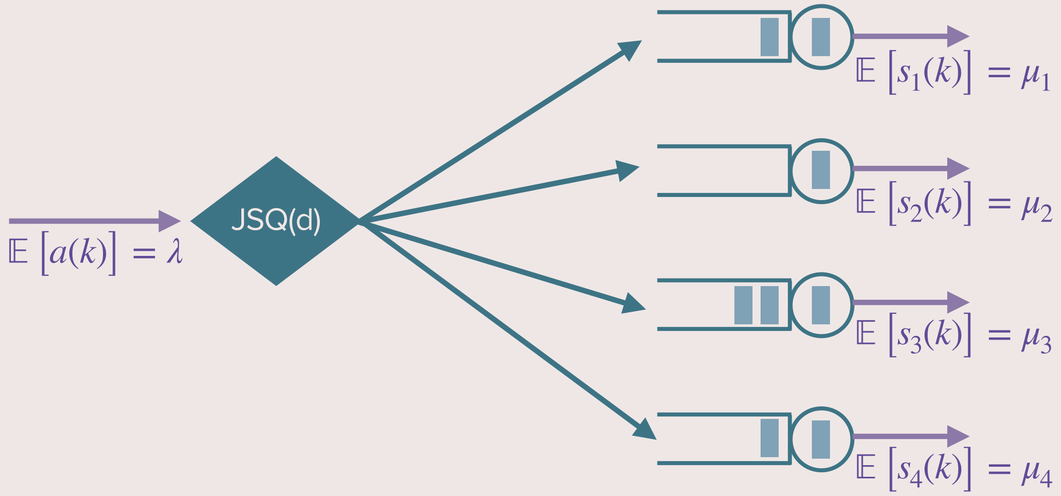

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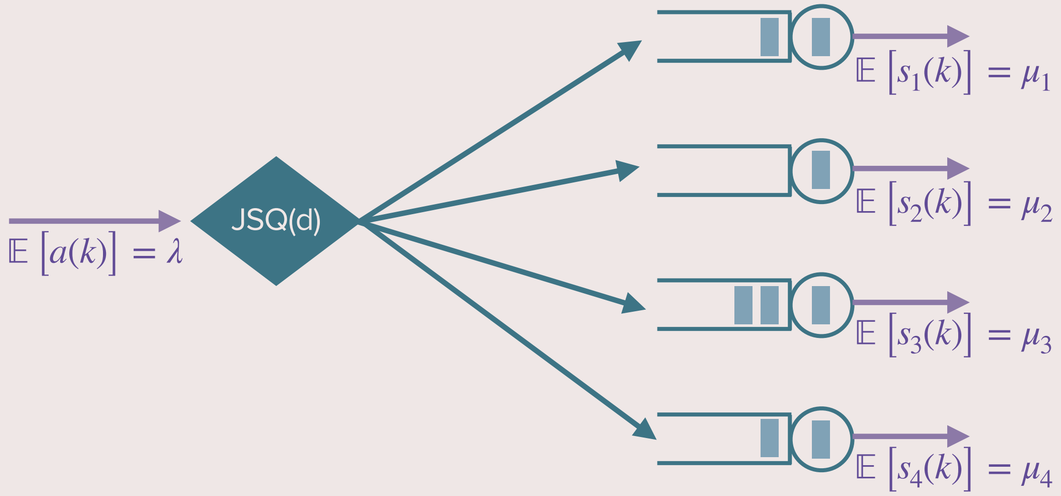

Then, the necessary and sufficient condition is that

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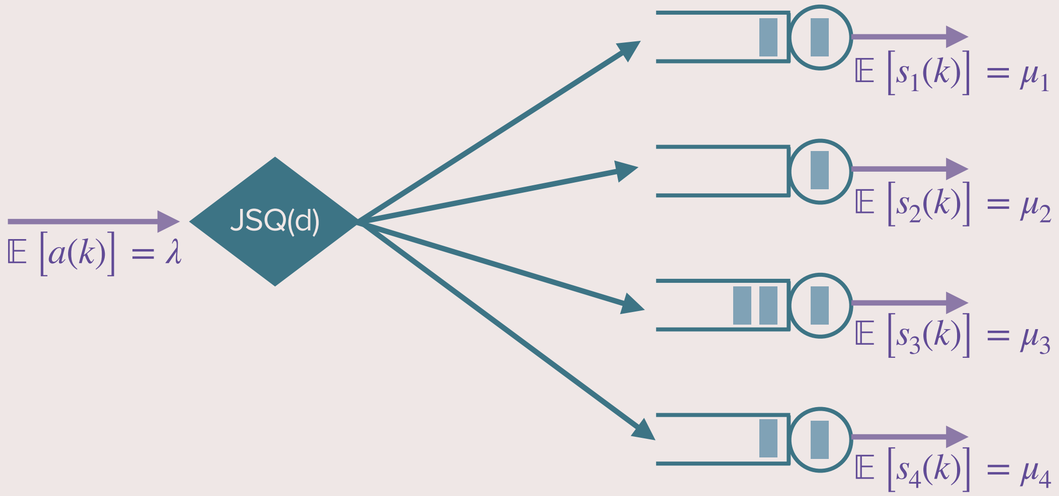

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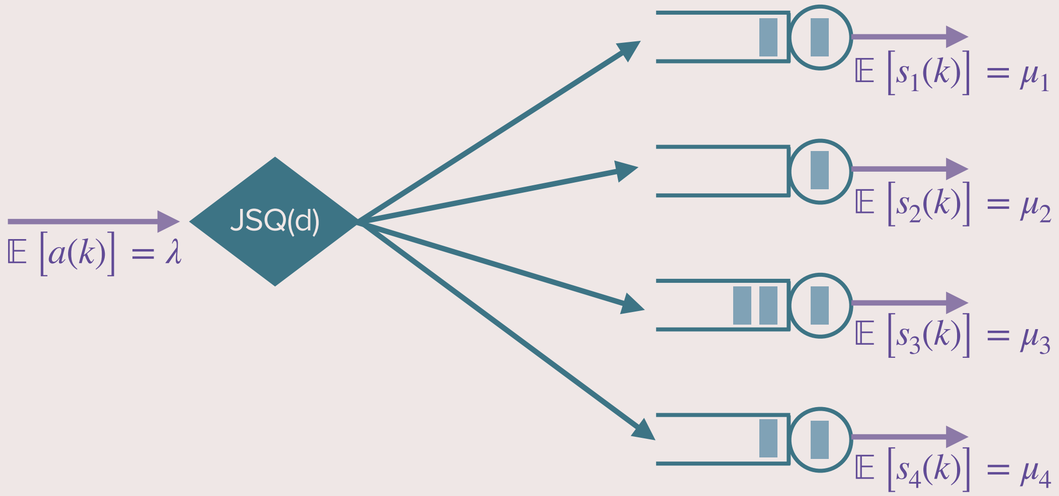

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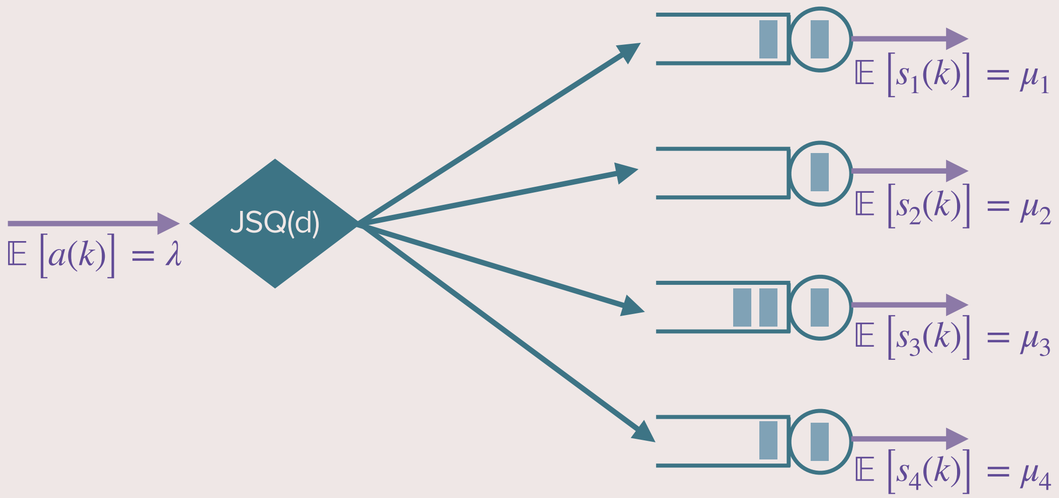

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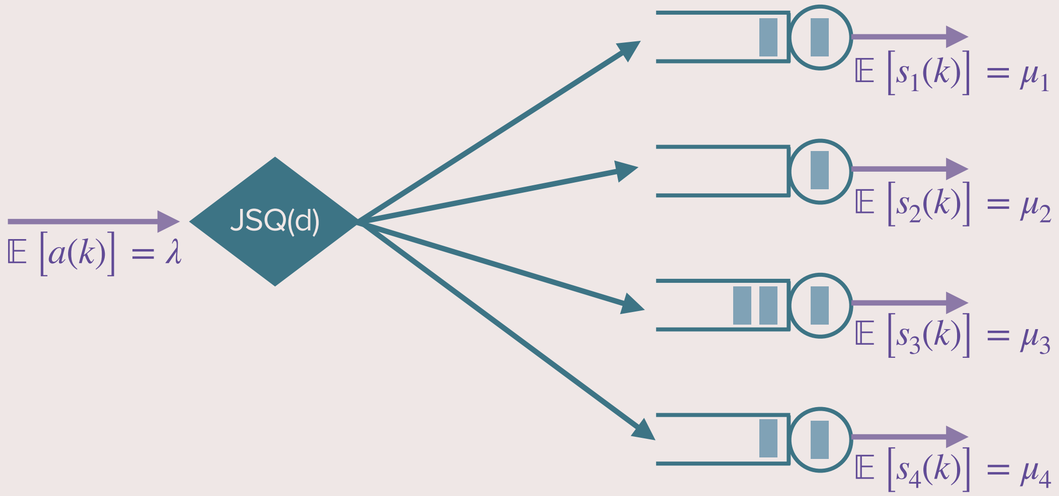

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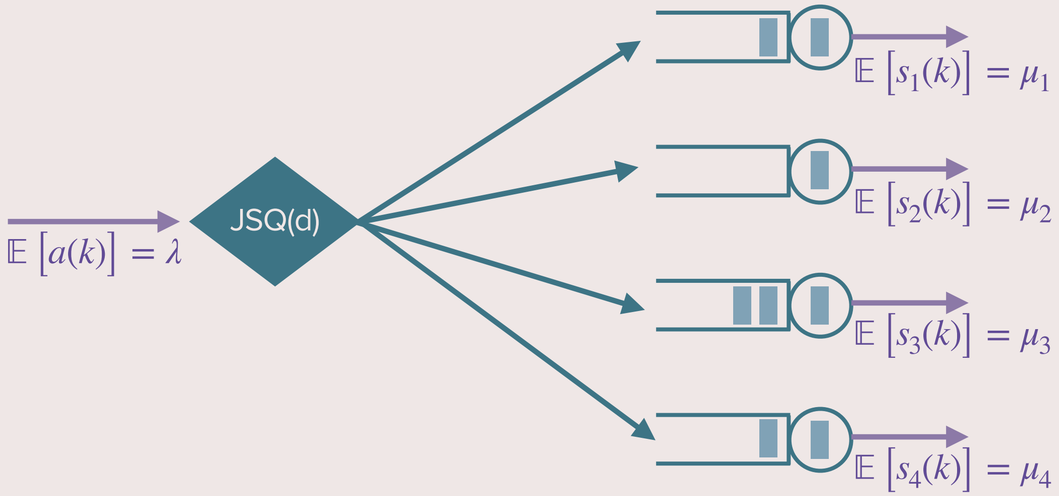

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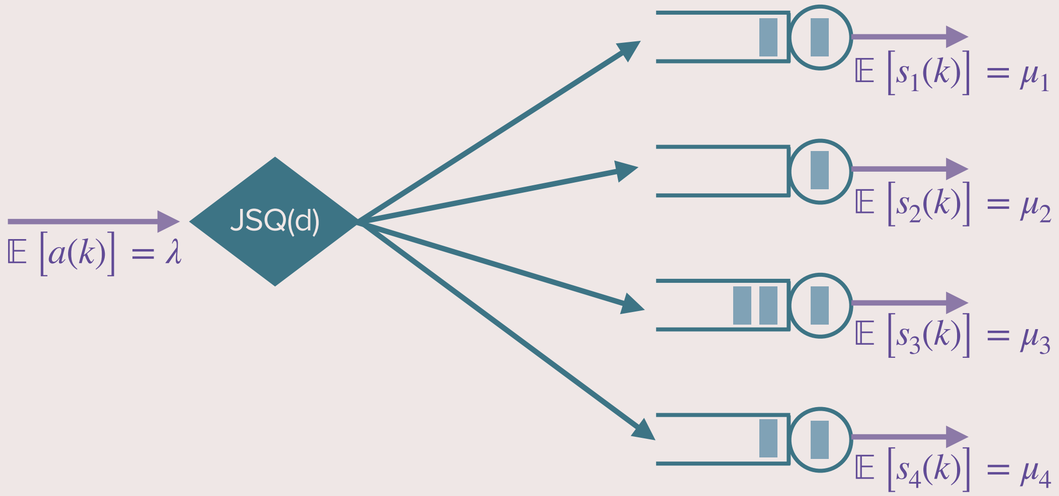

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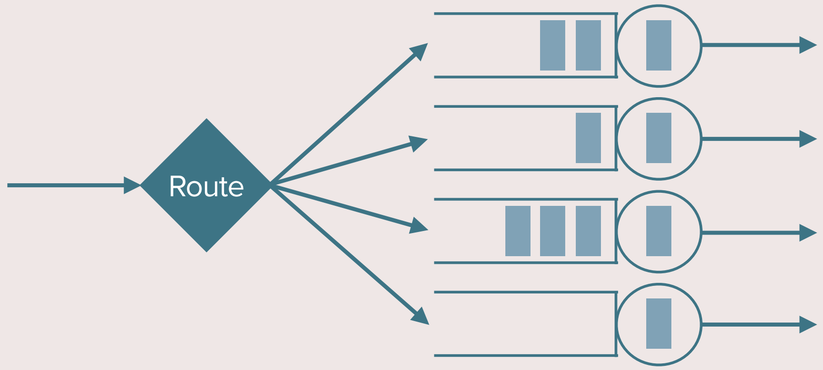
- Fix n. As d gets larger, we can tolerate more imbalance.

Theorem: JSQ(d) is throughput optimal if and only if

$$\frac{\sum_{i=1}^{j} \mu_{(i)}}{\mu_\Sigma} \geq \frac{\binom{j}{d}}{\binom{n}{d}} \qquad \forall d \leq j \leq n-1$$

Stability: Which One is Better?



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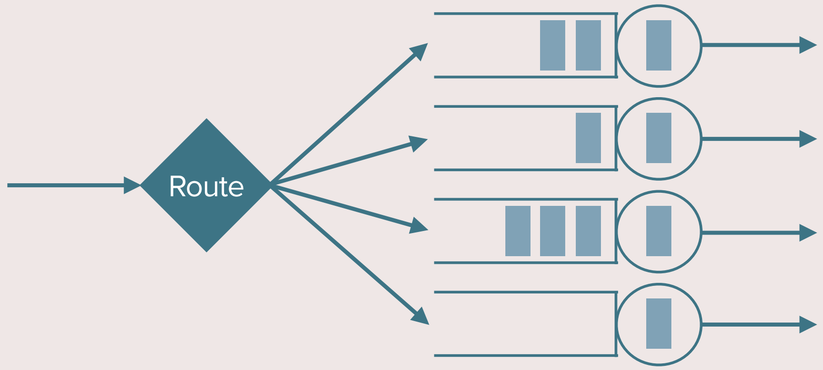
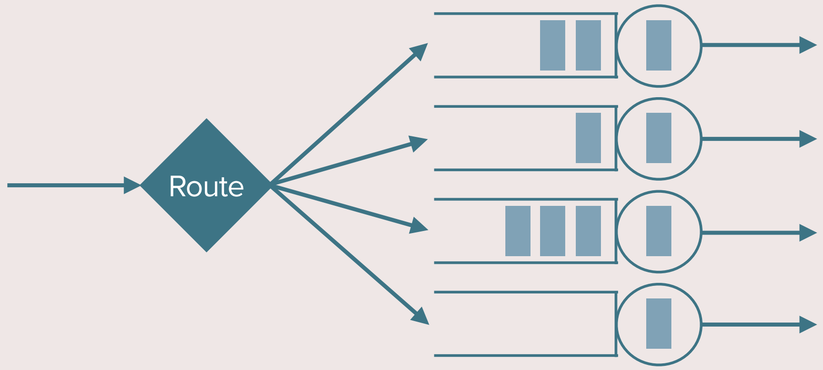


Random

JSQ(d)

JSQ

Stability: Which One is Better?



Random

Only if the service rates are equal

JSQ(d)

JSQ

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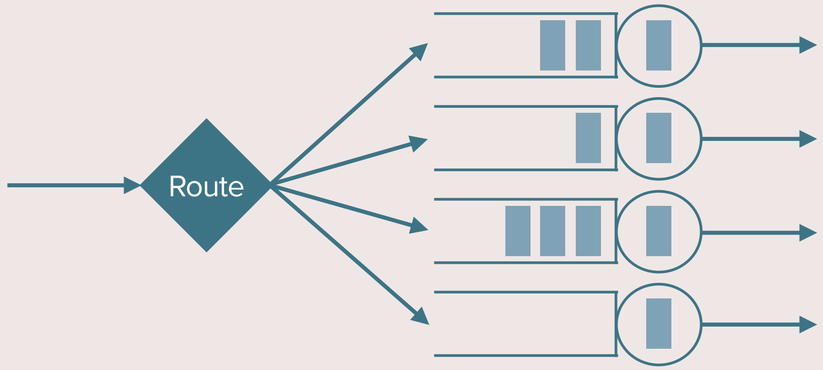

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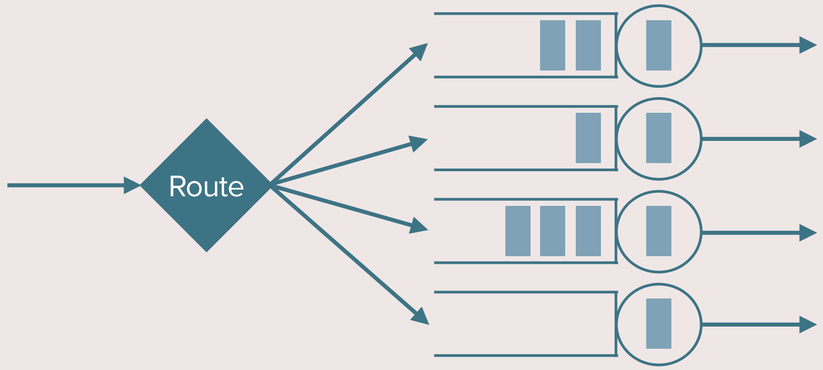
JSQ(d)

JSQ

All service rates

Stability: Which One is Better?



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Only if the service rates are equal

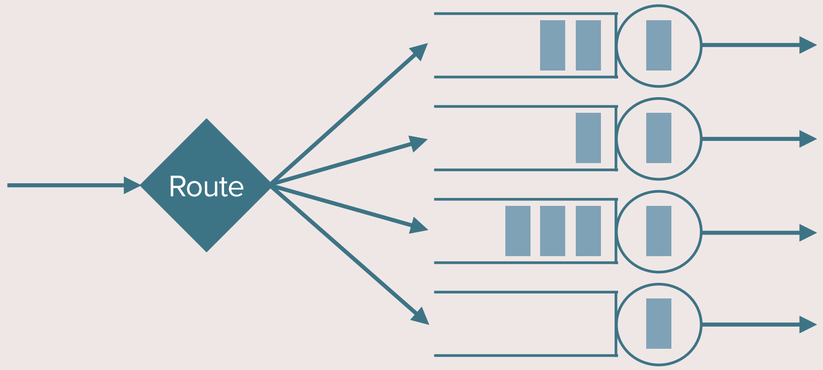
JSQ(d)

Majorization condition

JSQ

All service rates

Stability: Which One is Better?



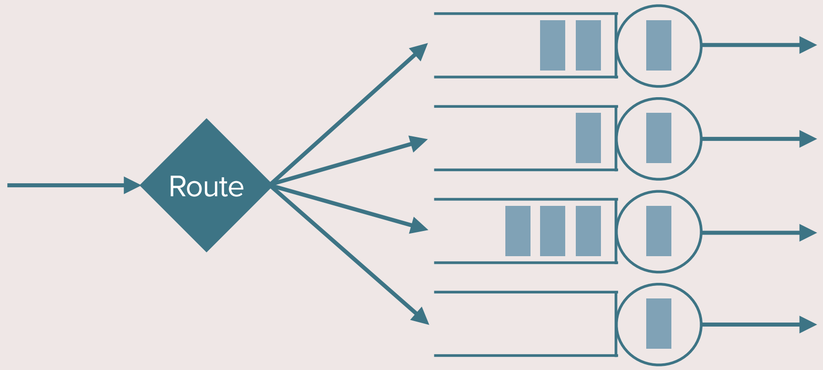
Random < **JSQ(d)** < **JSQ**

Only if the service rates are equal

Majorization condition

All service rates

Stability: Which One is Better?



Delay?

Asymptotic Analysis

Random < **JSQ(d)** < **JSQ**

Only if the service rates are equal

Majorization condition

All service rates

Heavy-Traffic Analysis

Heavy-Traffic Analysis

- Load the system close to maximum capacity:

Heavy-Traffic Analysis

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Arrival rate ≈ Service rate

Heavy-Traffic Analysis

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- State Space Collapse (SSC)

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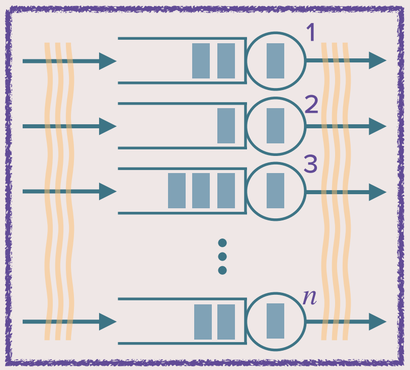

Heavy-Traffic Analysis

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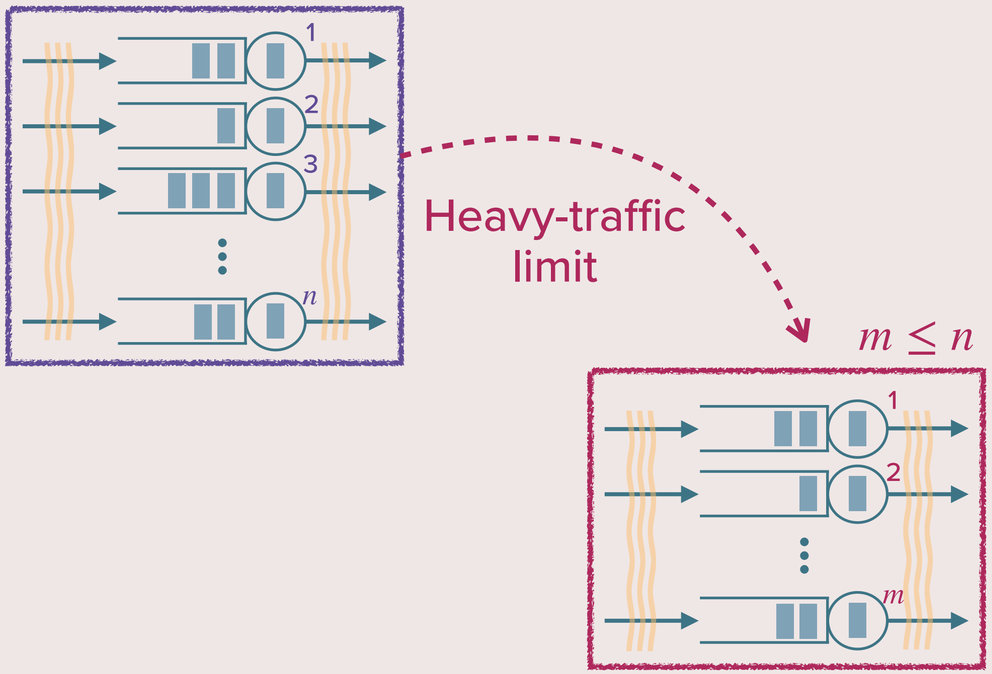

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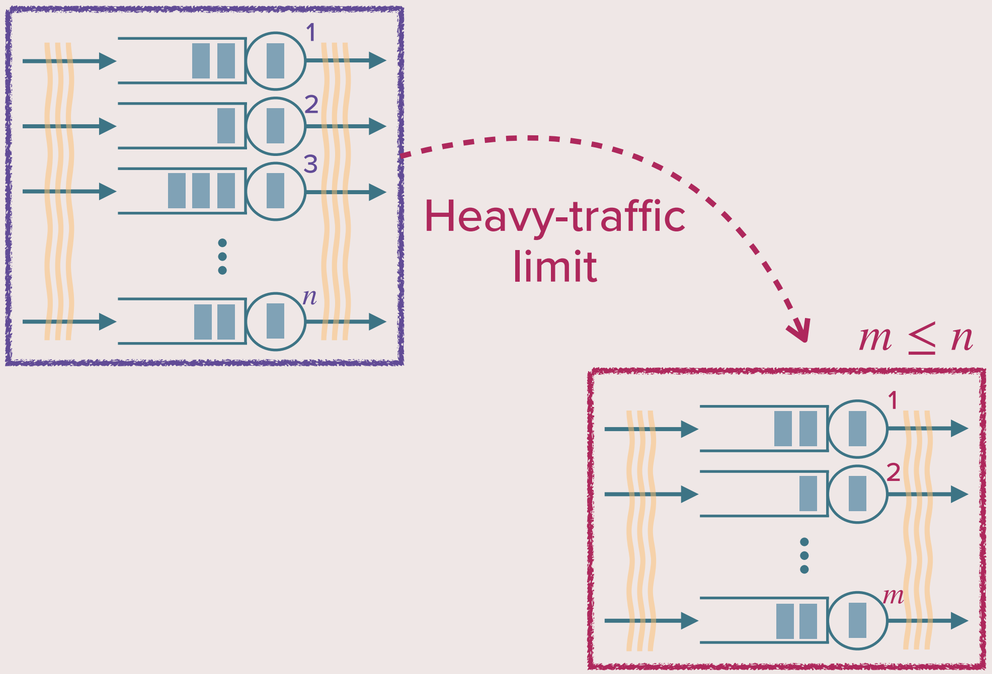
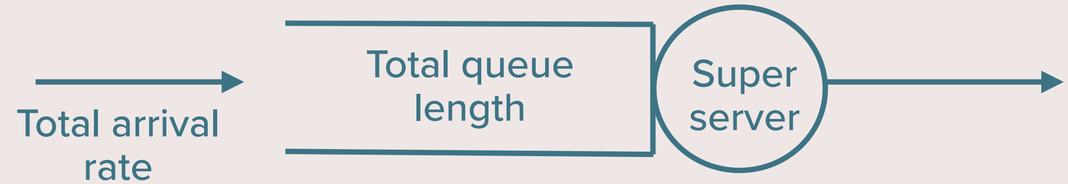


Complete Resource Pooling (CRP):

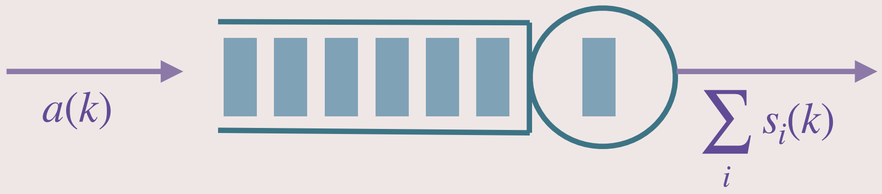
- SSC to a one-dimensional subspace
- System behaves as a single server queue
- All servers “pool” together and behave as a super server



Importance of CRP

Importance of CRP

CRP



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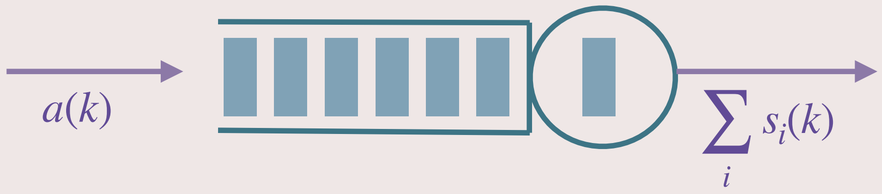

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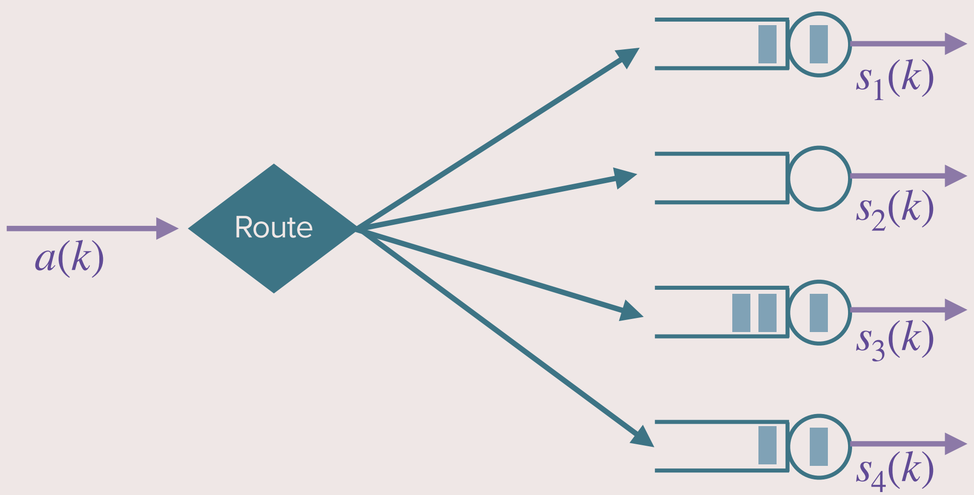

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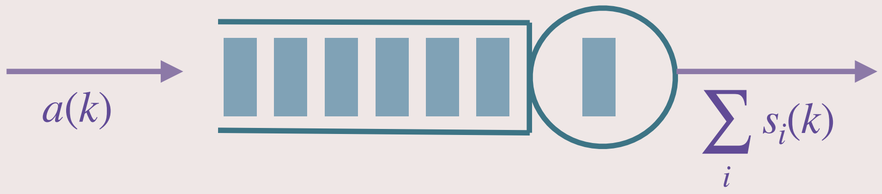

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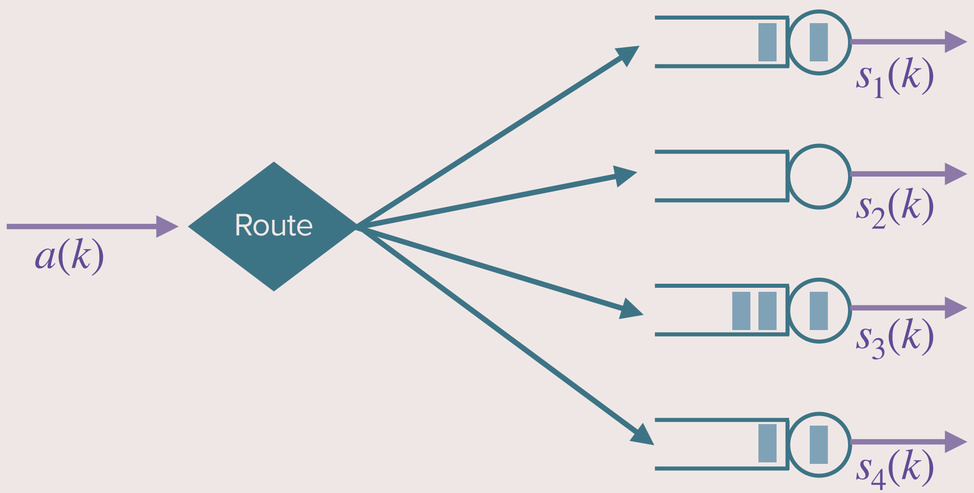
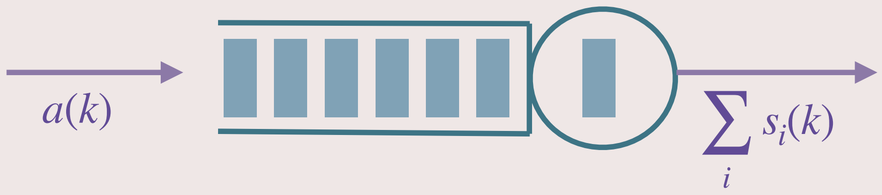


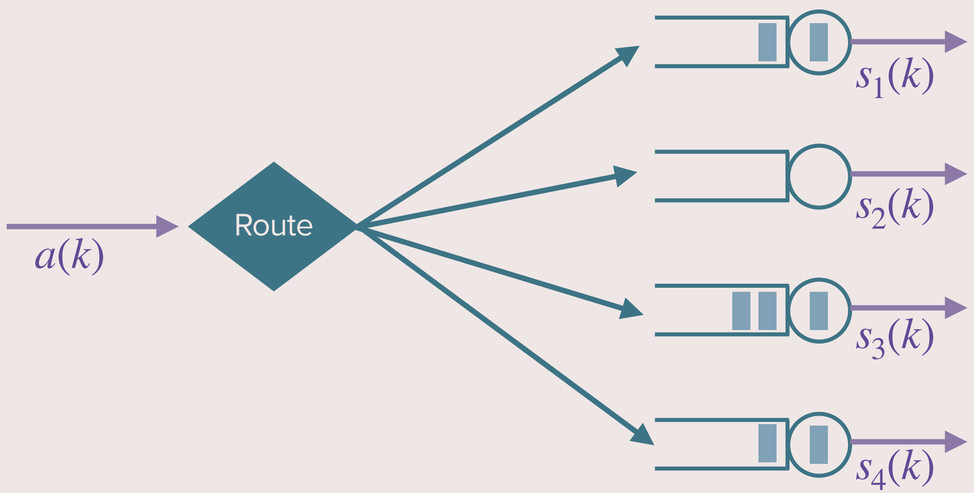
- To do: 7 jobs

Importance of CRP

CRP



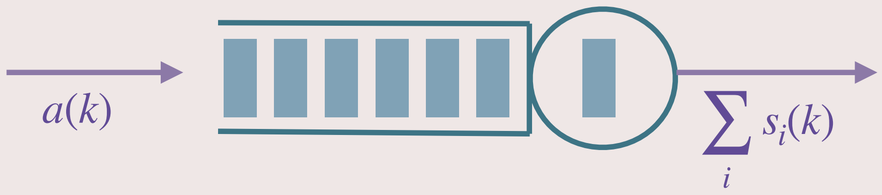
No CRP



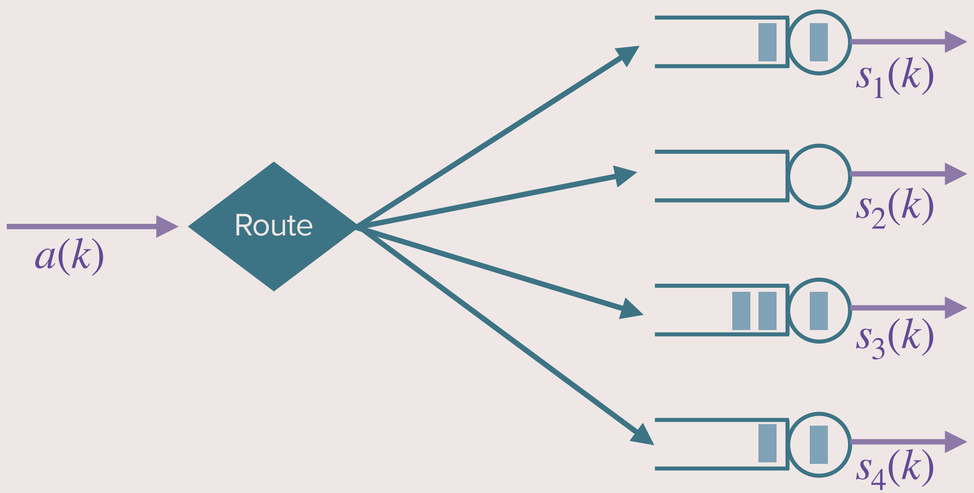
- To do: 7 jobs
- Same total service capacity

Importance of CRP

CRP



No CRP



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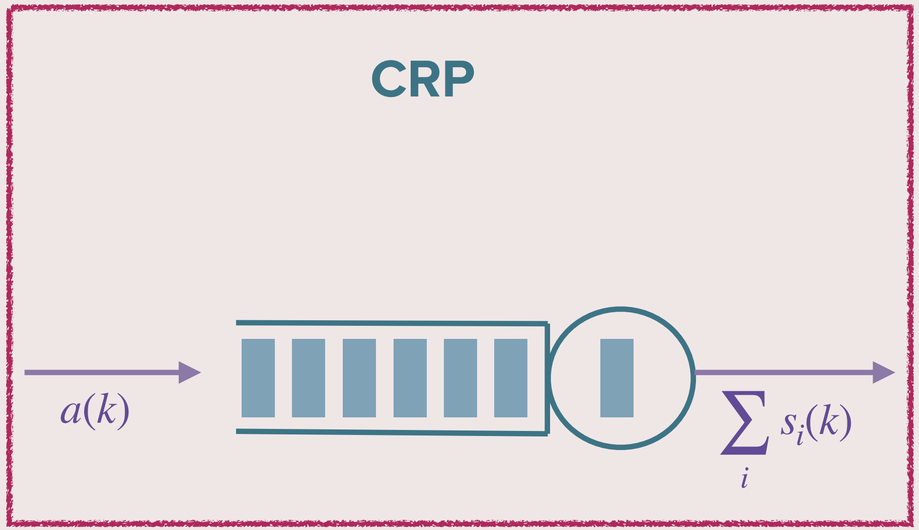
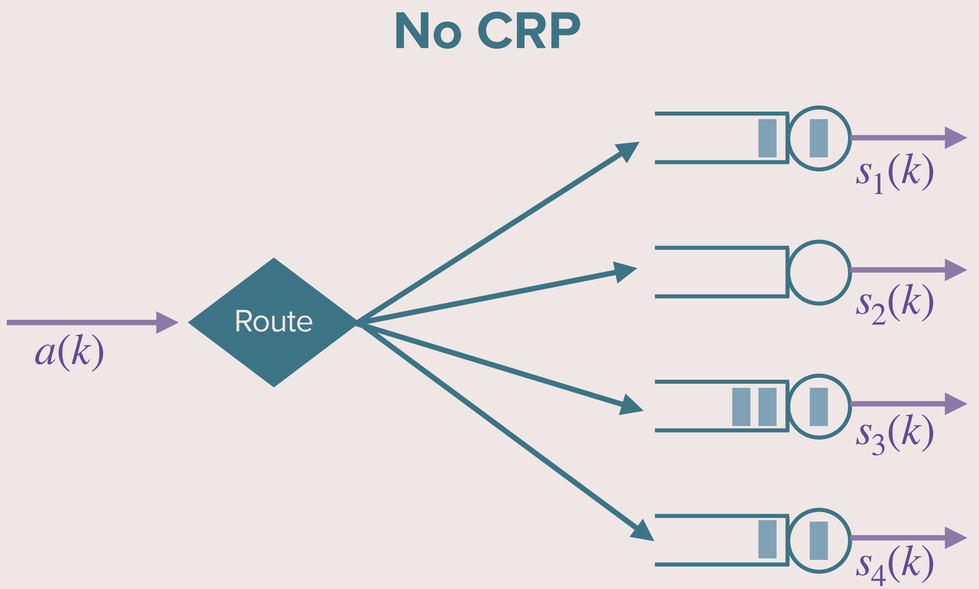
Which one finishes first?

Importance of CRP





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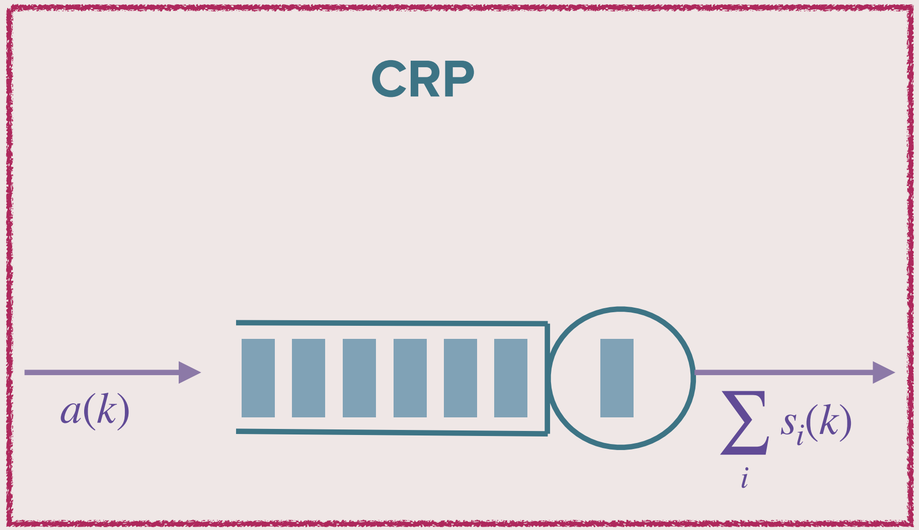

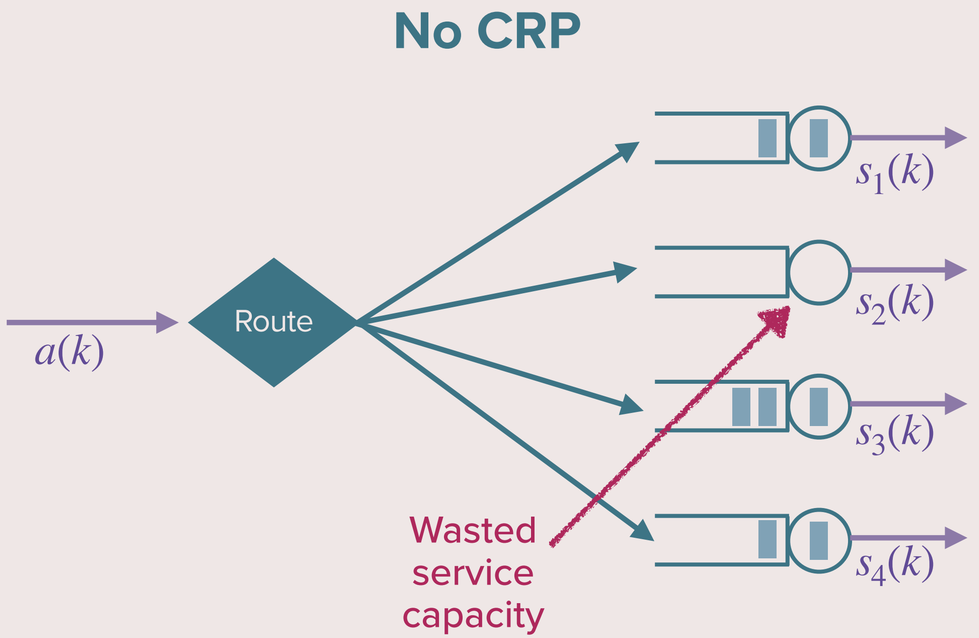

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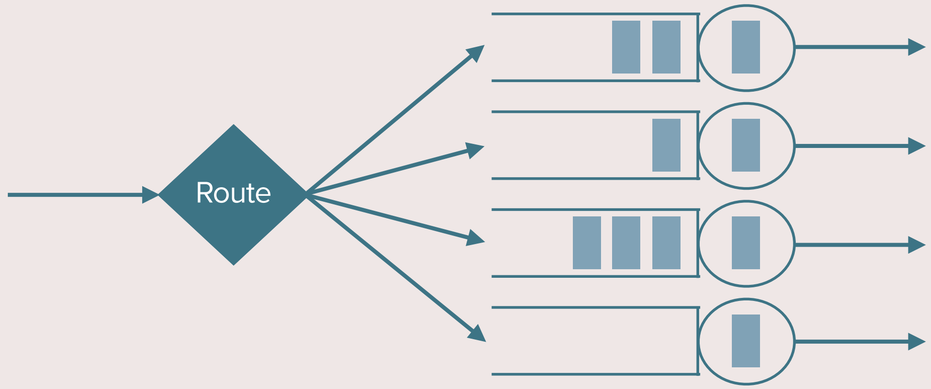
CRP in Our Routing Algorithms



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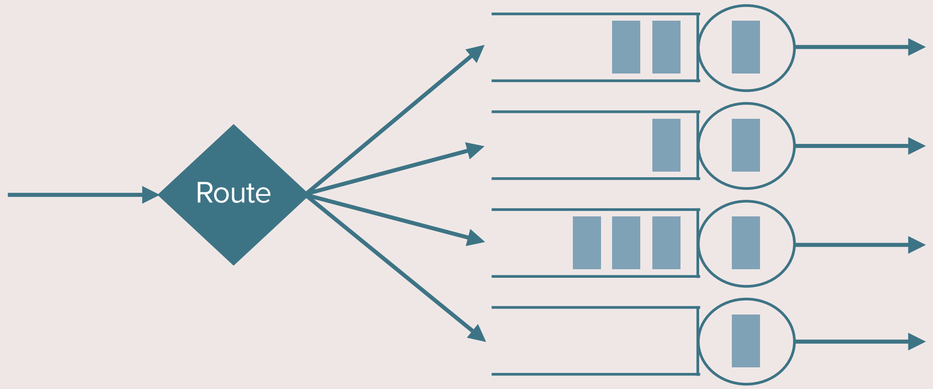
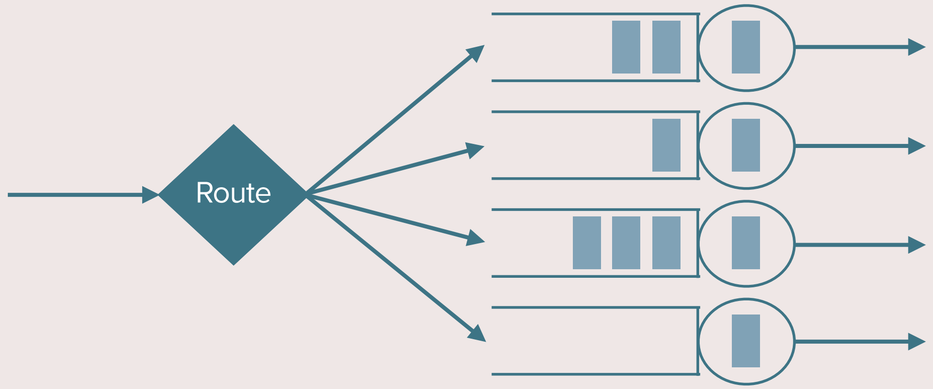


Random

JSQ(d)

JSQ

CRP in Our Routing Algorithms



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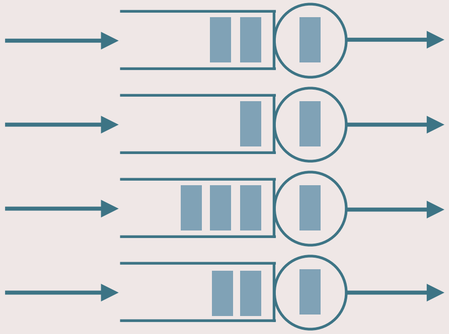

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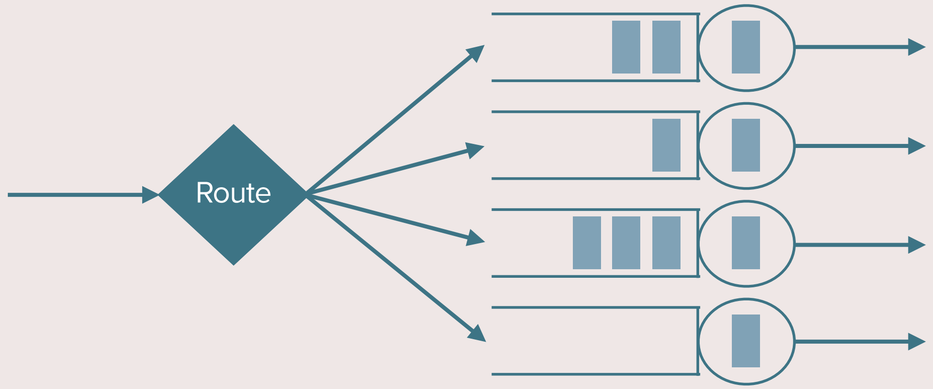

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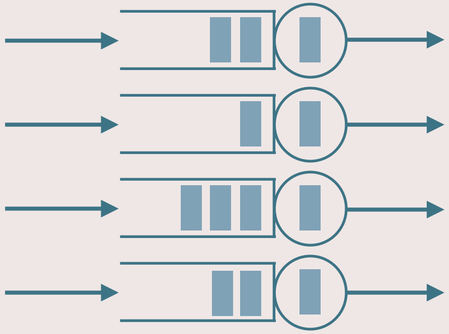

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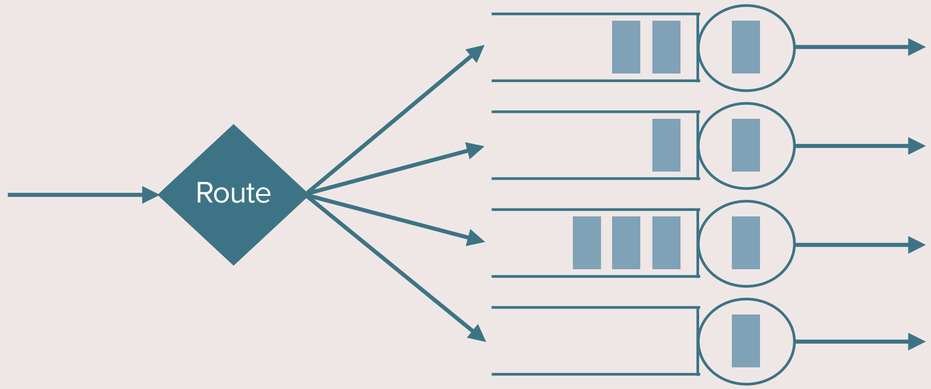

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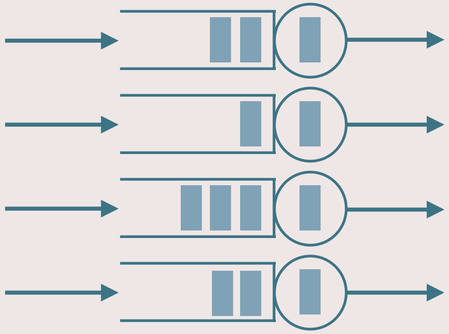


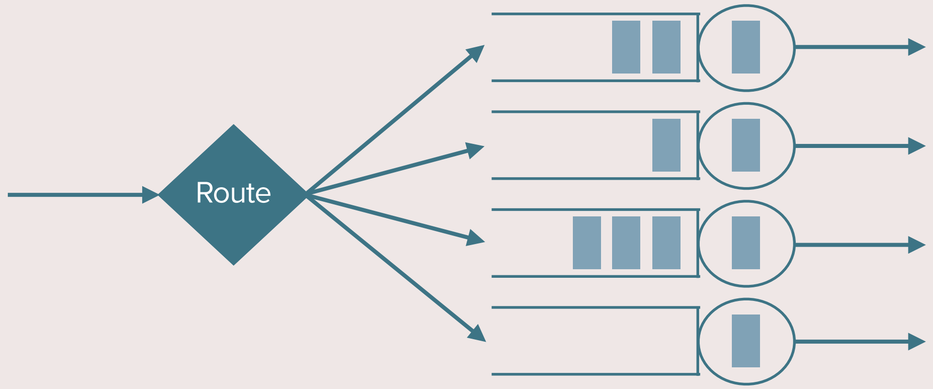
No CRP

JSQ(d)

JSQ

Tries to make all queues equal

CRP in Our Routing Algorithms



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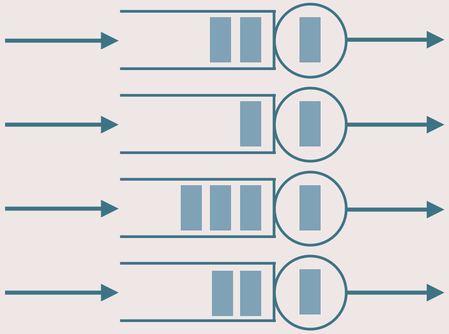

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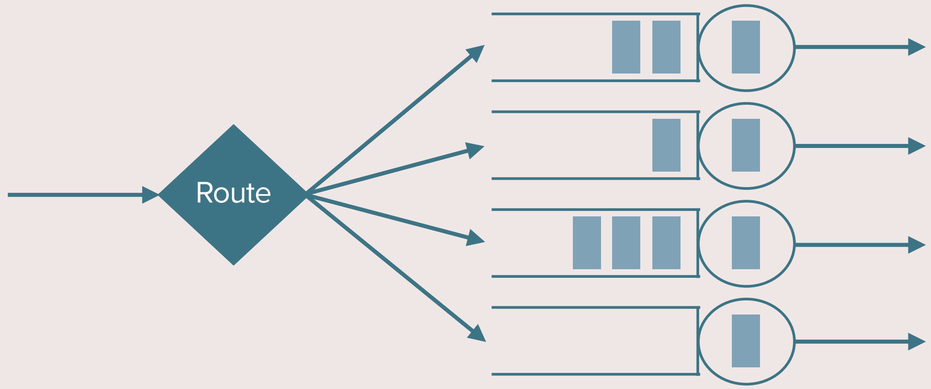

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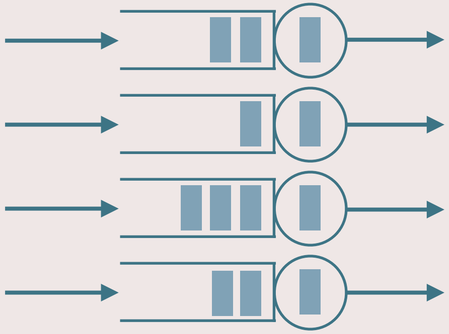


No CRP

JSQ(d)

Select queues at random

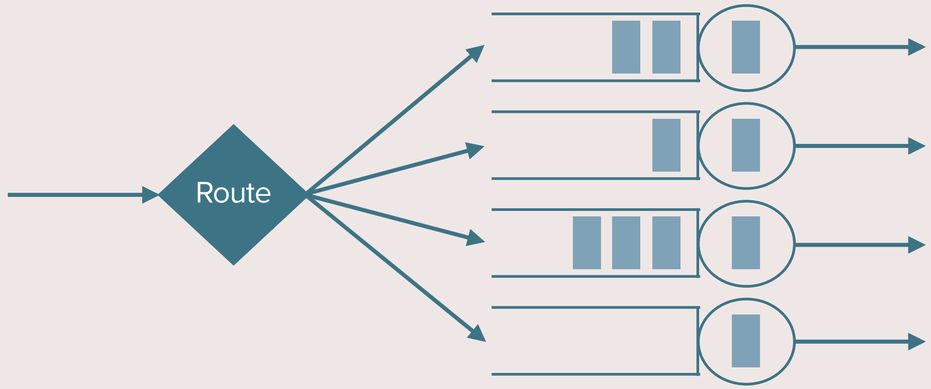
Route JSQ

JSQ

Tries to make all queues equal

CRP

CRP in Our Routing Algorithms



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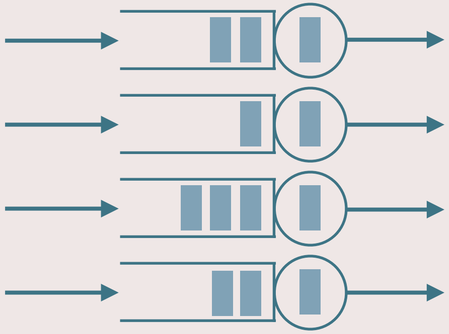

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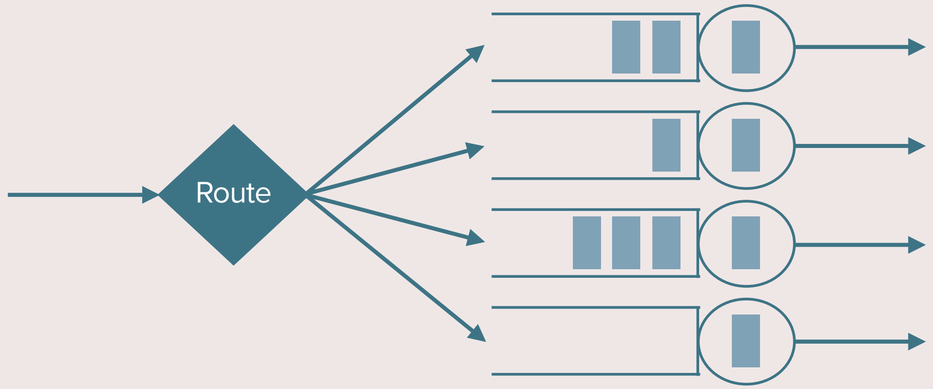

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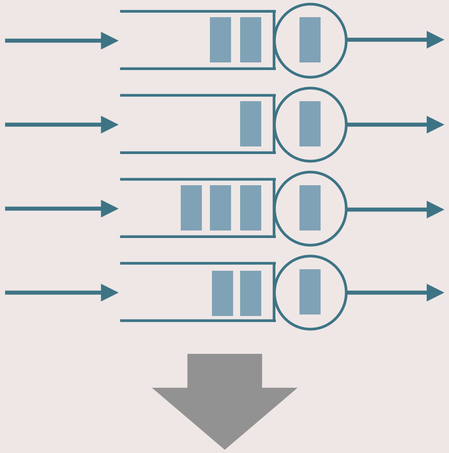

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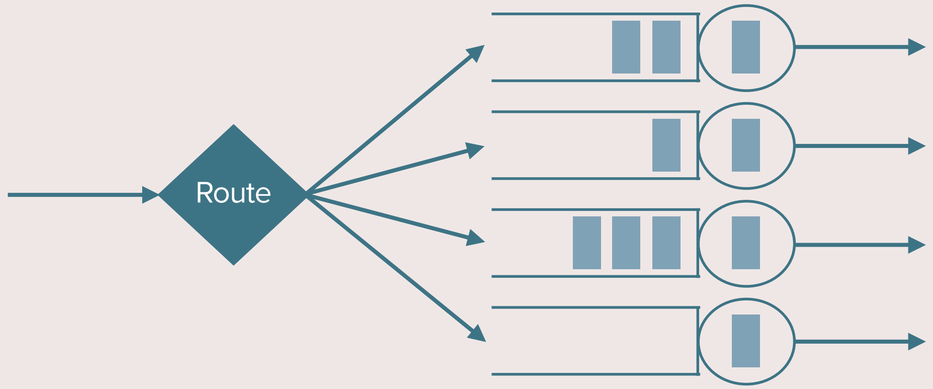

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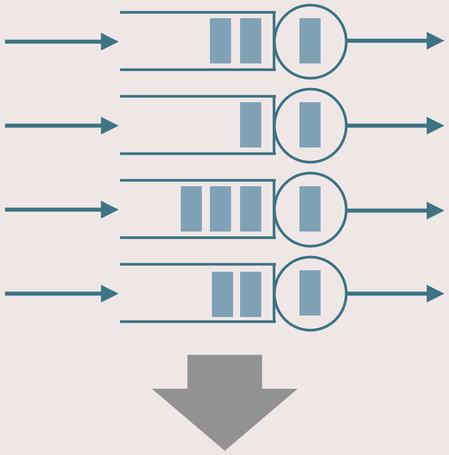

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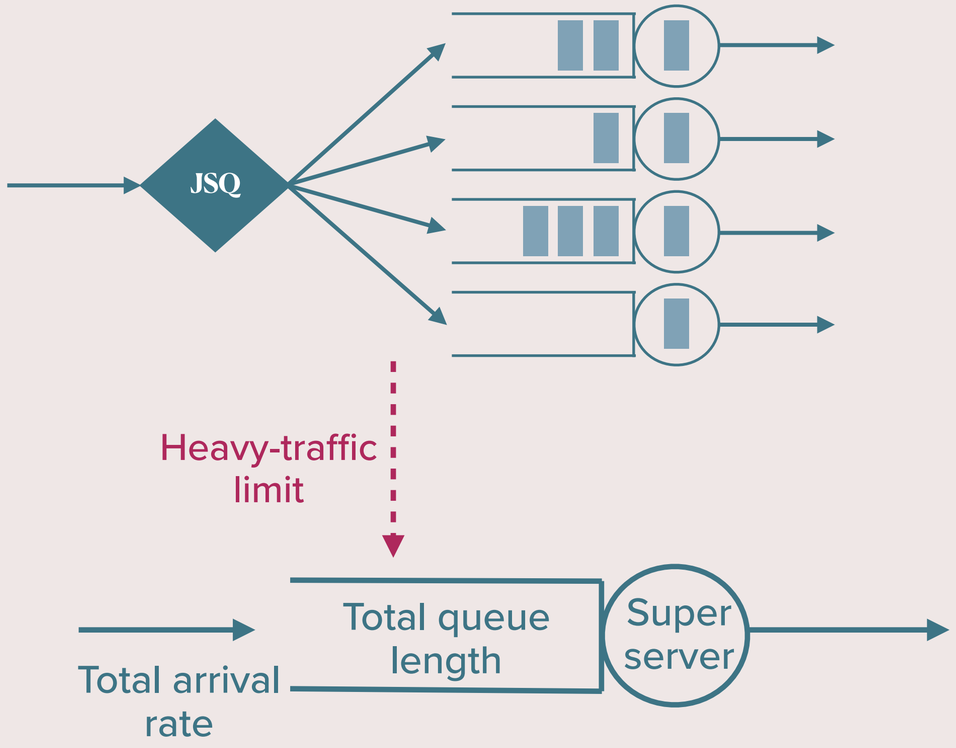
In the Literature



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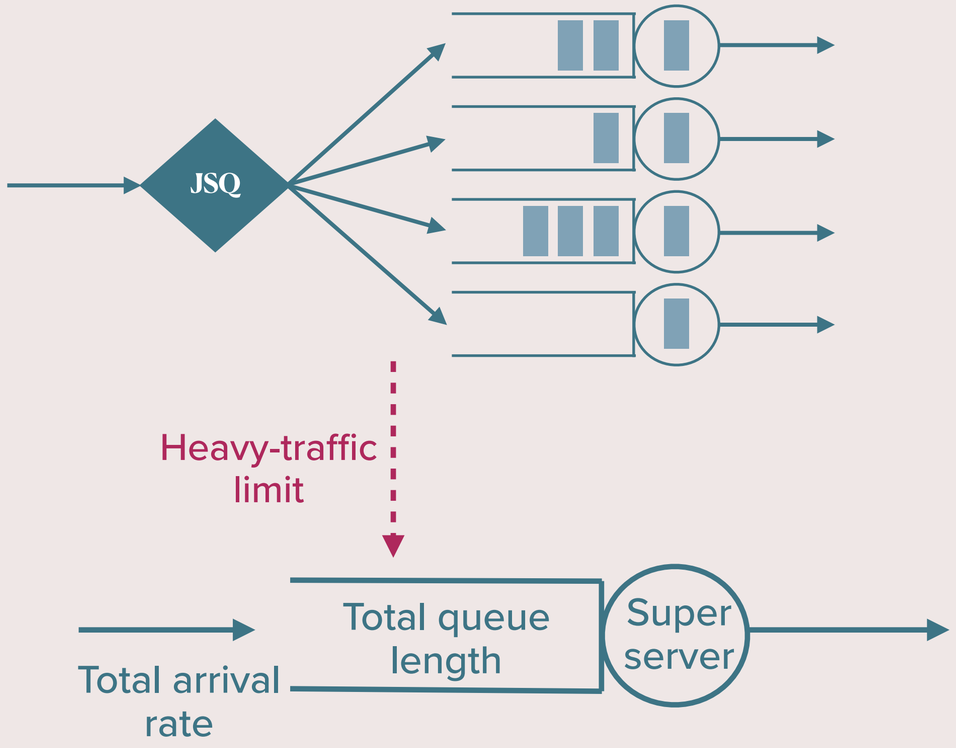

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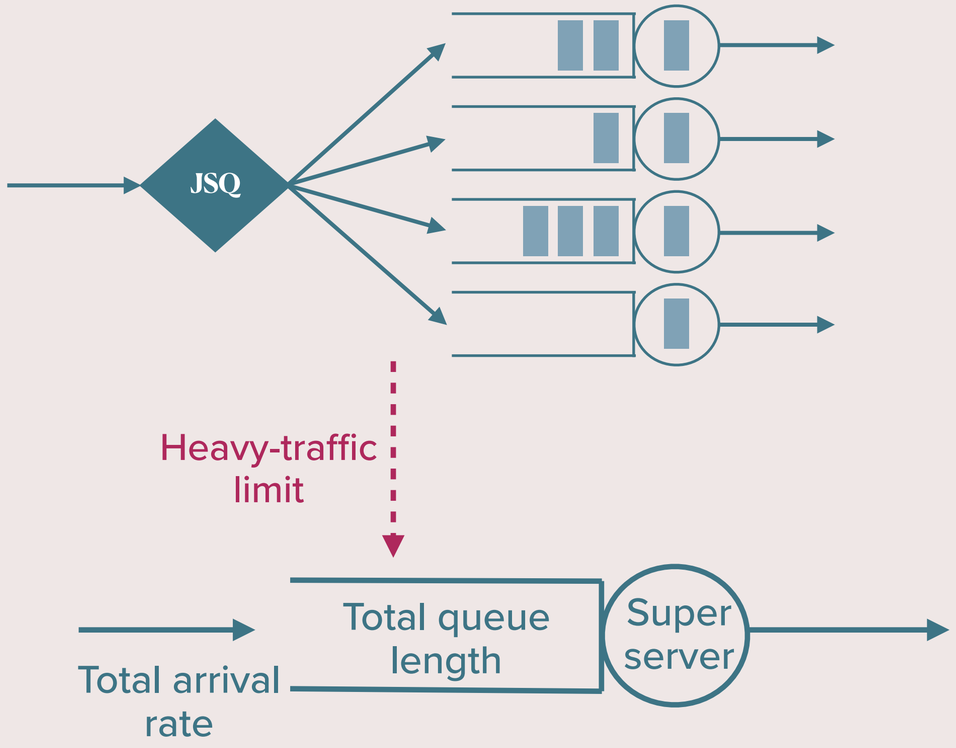

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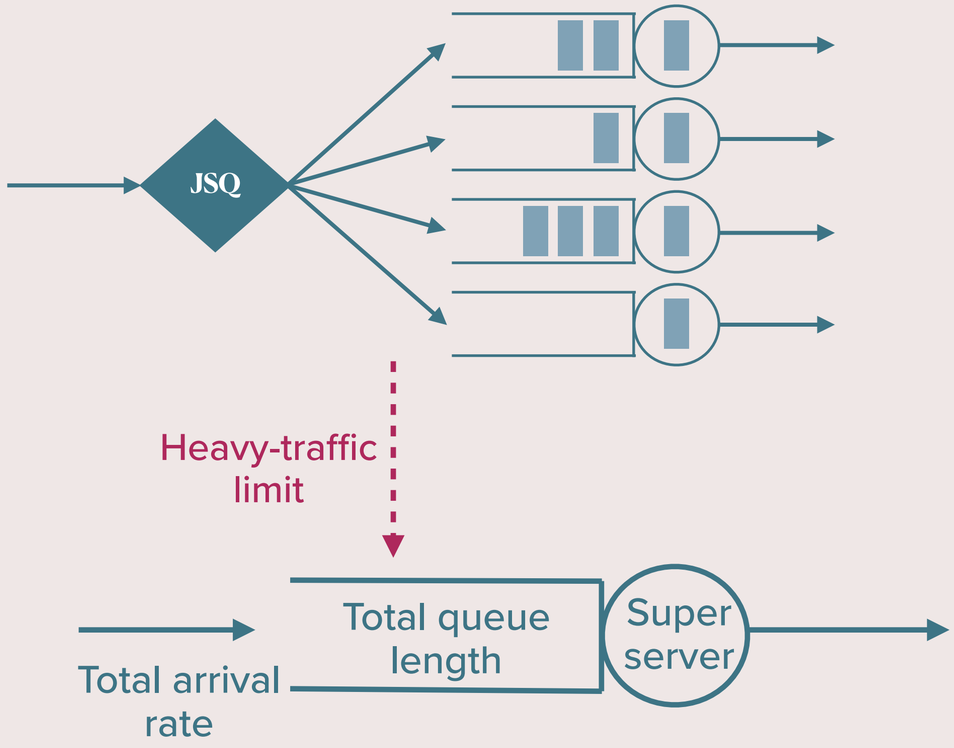

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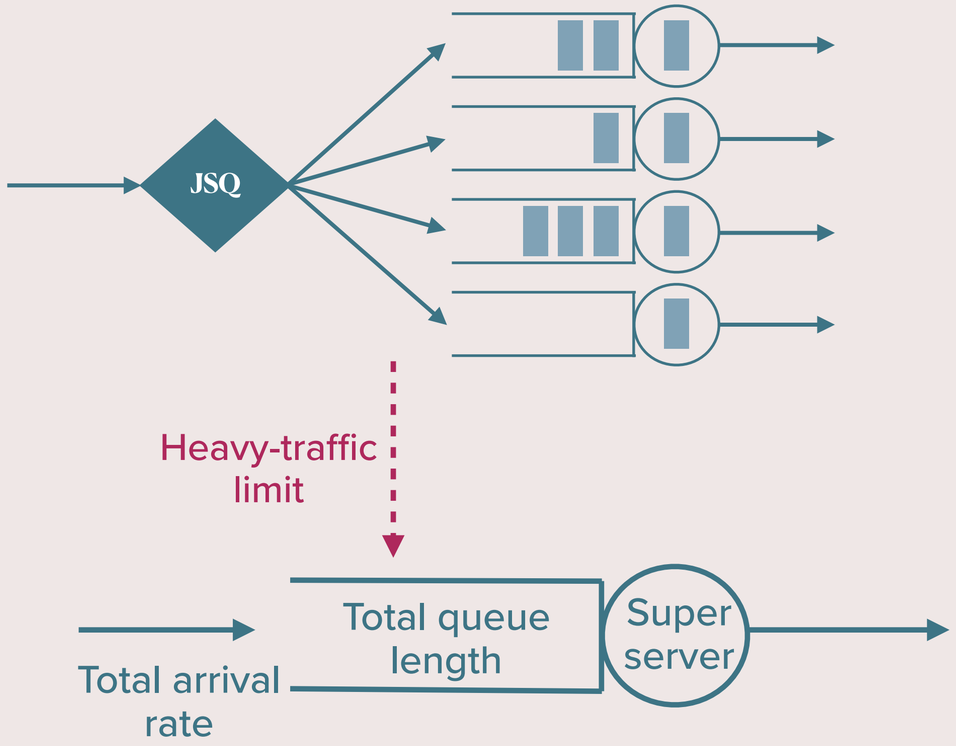

Diffusion limits approach

- Most popular
- Introduced by Kingman (1962)
- Show convergence in distribution of queue length scaled heavy-traffic parameter ϵ

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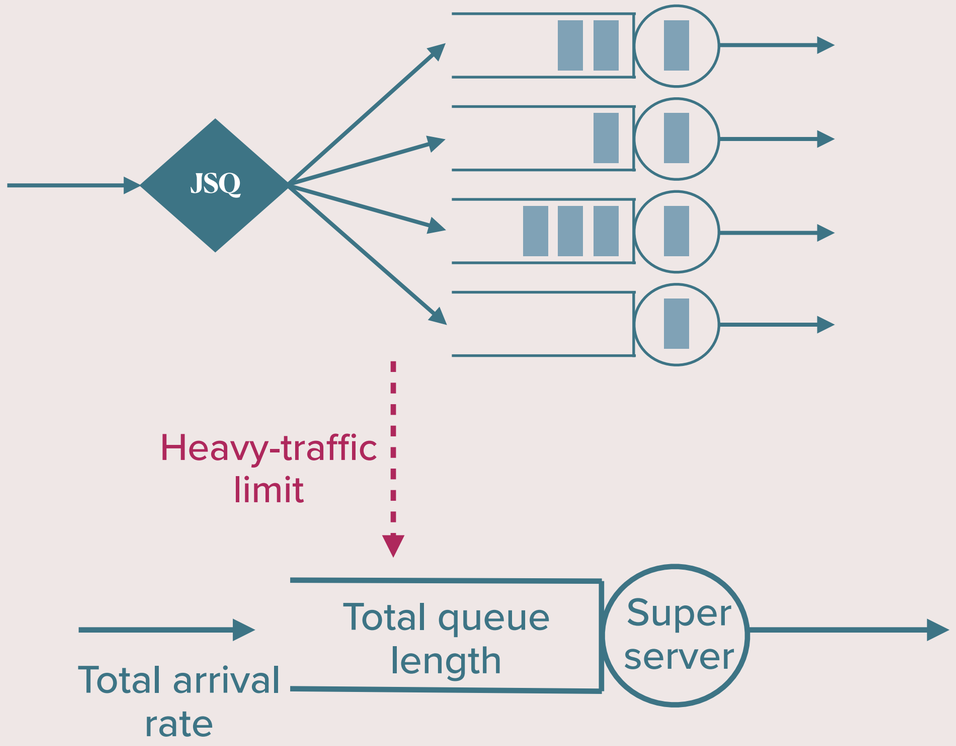

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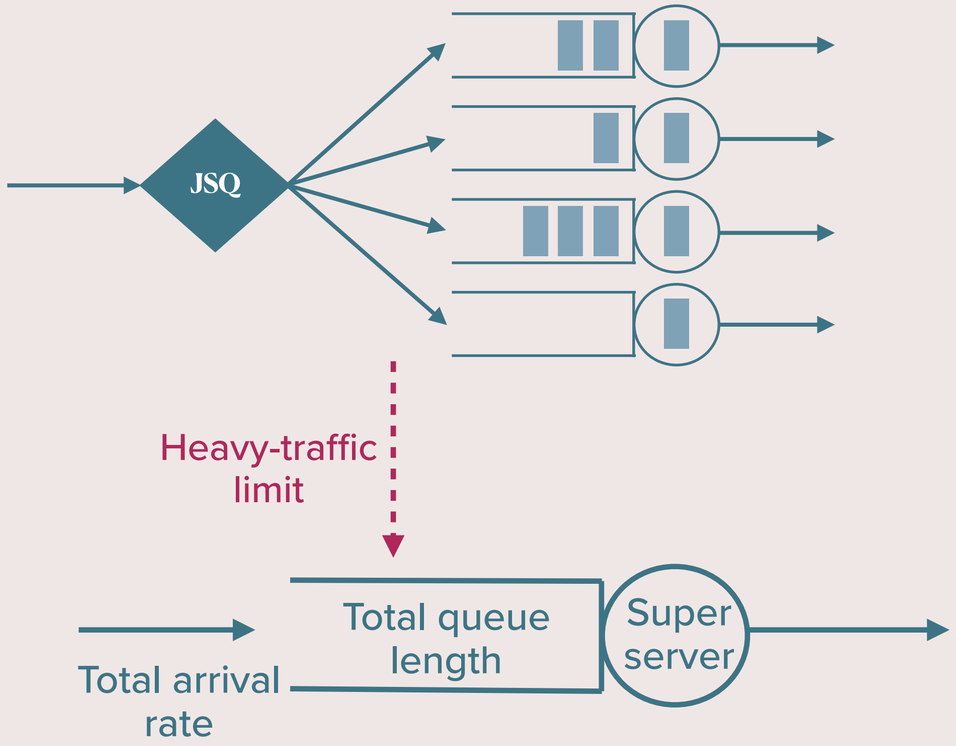

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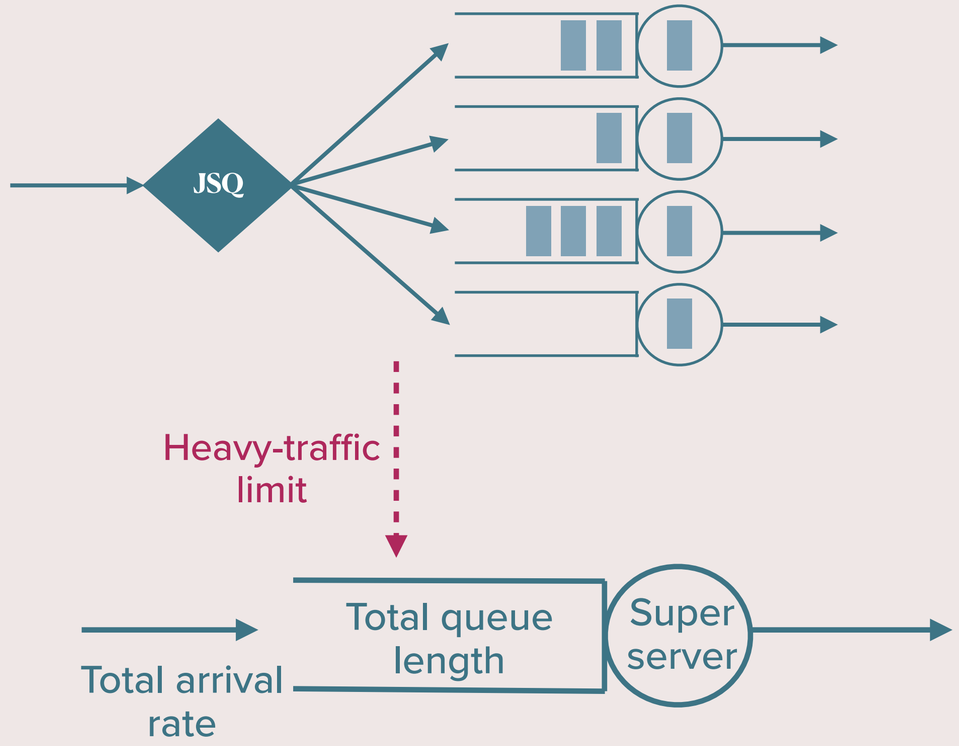

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RBM

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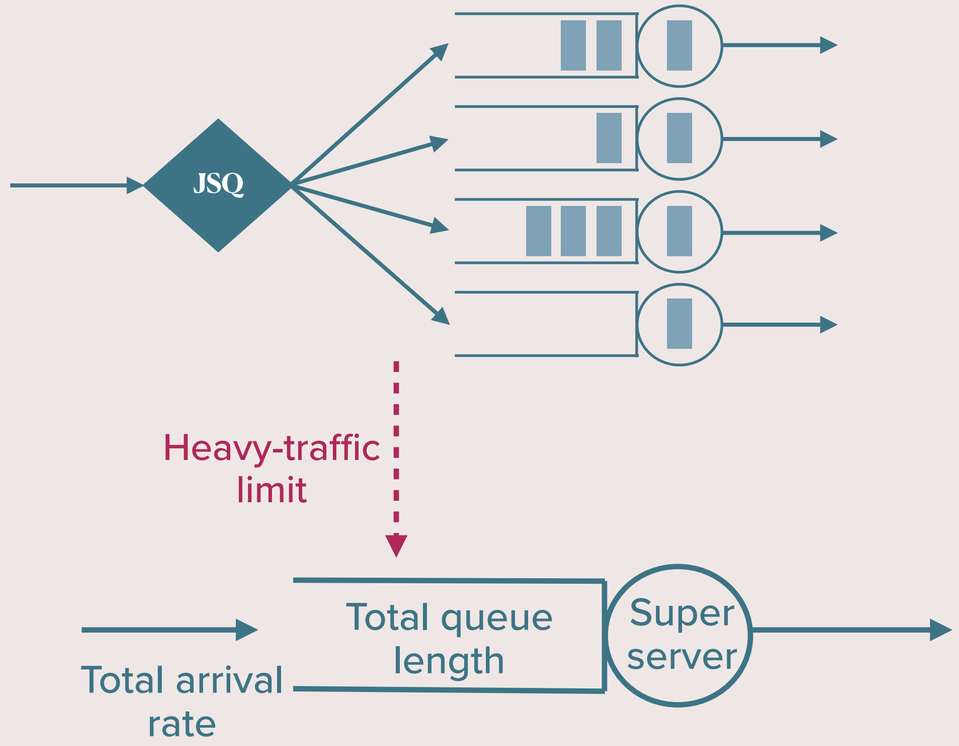

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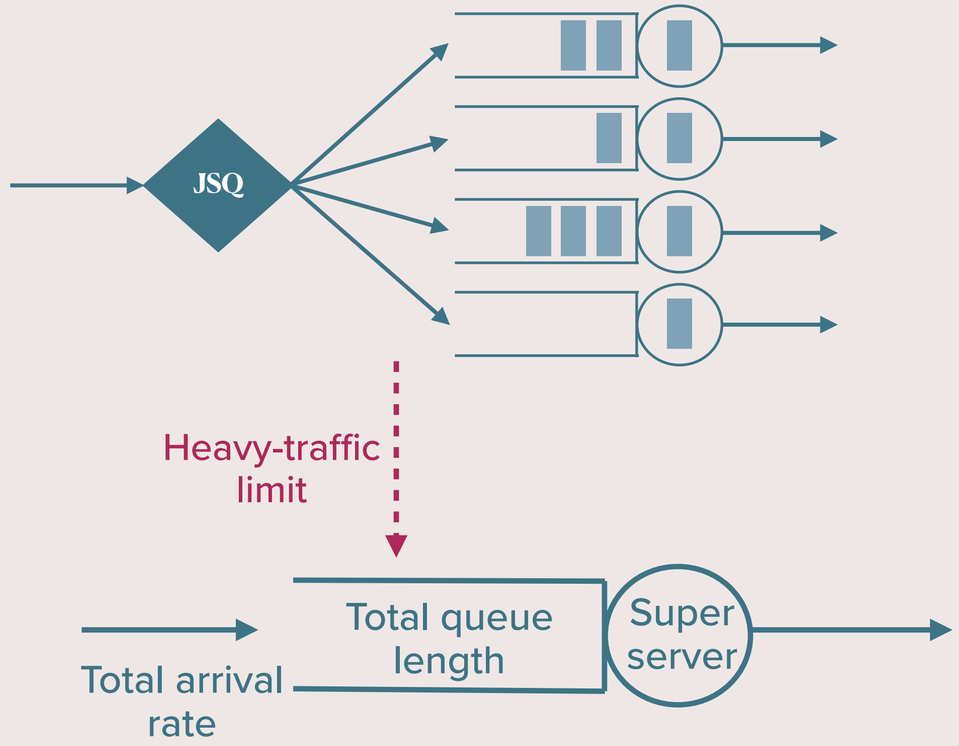

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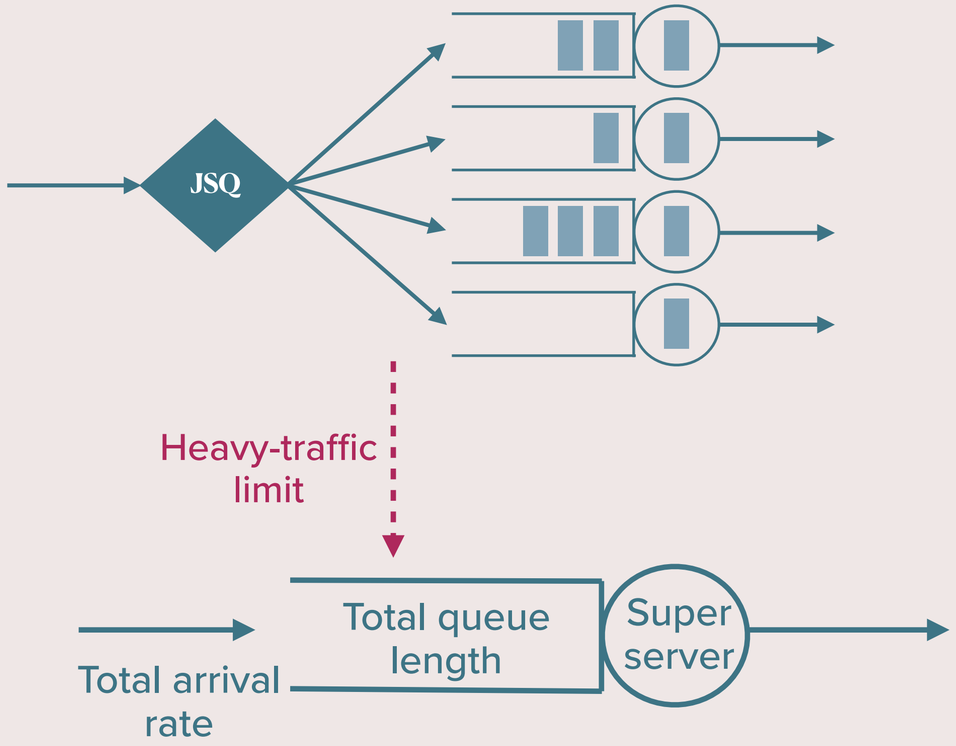

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RBM

We propose a proof in 4 lines

The Single-Server Queue

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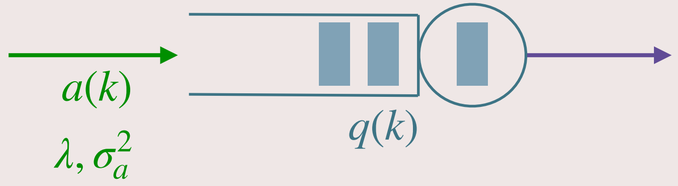

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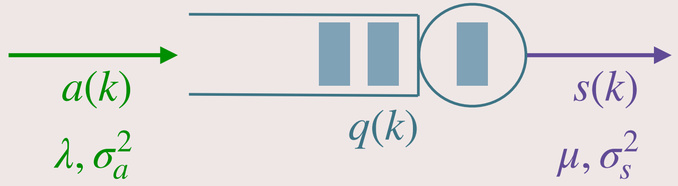

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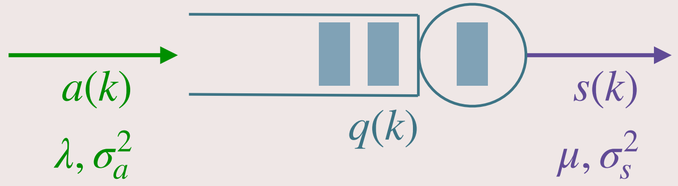

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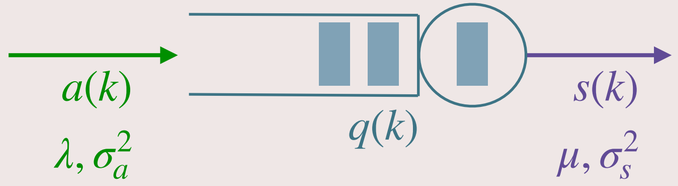

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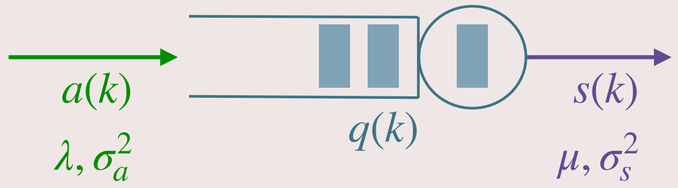

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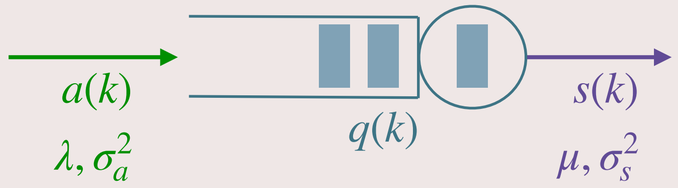

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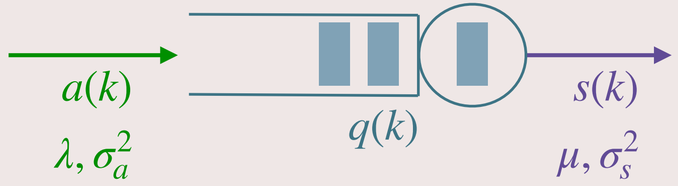

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We wish to show:

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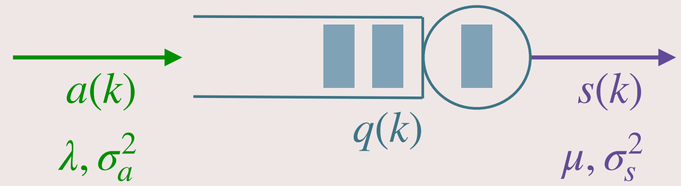
The MGF Method [HL, Maguluri '20]



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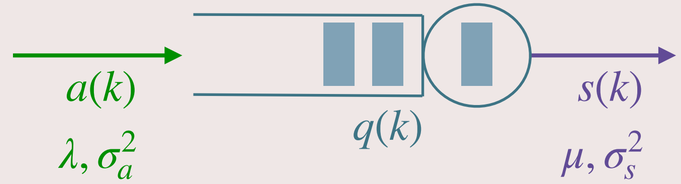

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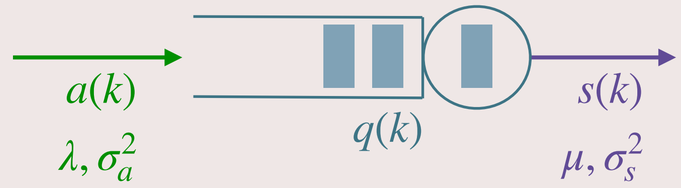

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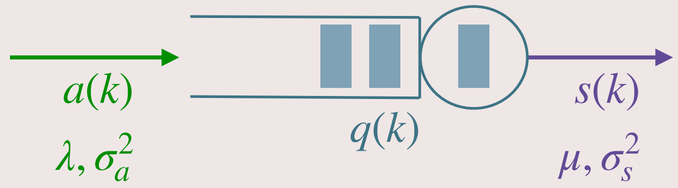

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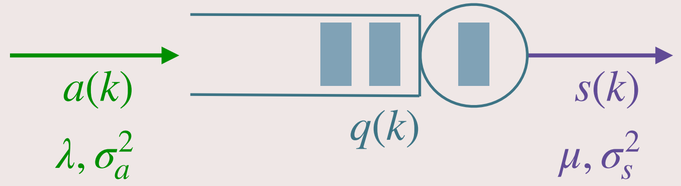

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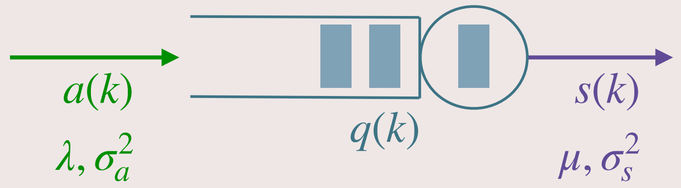

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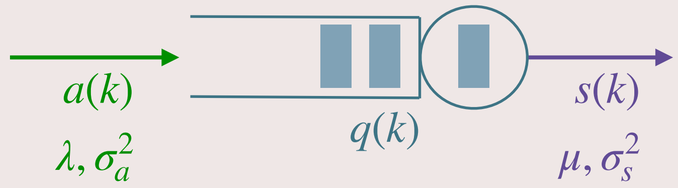

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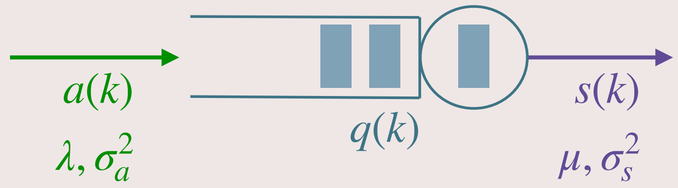

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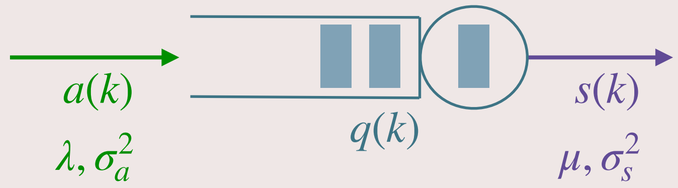

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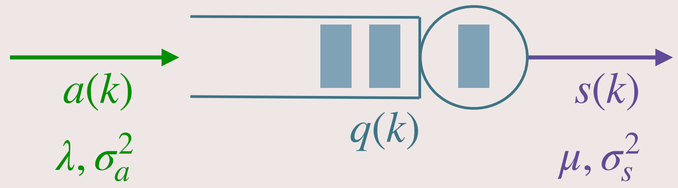

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MGF of expo. random variable with mean $\dfrac{\sigma_a^2 + \sigma_s^2}{2}$

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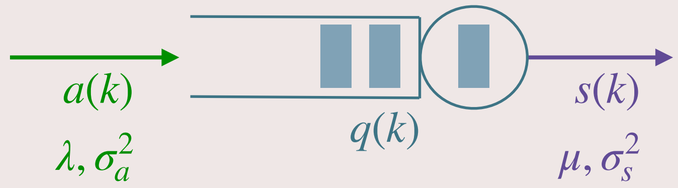

Dynamics of the queues:
$$q(k+1) = q(k) + a(k) - s(k) + u(k)$$
$$\Longrightarrow q(k+1)u(k) = 0$$

- From the lemma:

$$\mathbb{E}\left[e^{\theta\epsilon q(k+1)}\right] = \mathbb{E}\left[e^{\theta\epsilon(q(k)+a(k)-s(k))} - e^{-\theta\epsilon u(k)} + 1\right]$$

$$\Longrightarrow \mathbb{E}\left[e^{\theta\epsilon q}\right] = \frac{1 - \mathbb{E}\left[e^{-\theta\epsilon u}\right]}{1 - \mathbb{E}\left[e^{\theta\epsilon(a-s)}\right]}$$

Step 2: Bound unused service and take the heavy-traffic limit

$$\Longrightarrow \lim_{\epsilon\downarrow 0} \mathbb{E}\left[e^{\theta\epsilon q}\right] = \frac{1}{1 - \theta\left(\frac{\sigma_a^2 + \sigma_s^2}{2}\right)}$$

MGF of expo. random variable with mean $\dfrac{\sigma_a^2 + \sigma_s^2}{2}$

$$\epsilon q \Rightarrow Expo\left(\frac{2}{\sigma_a^2 + \sigma_s^2}\right)$$

Transform Techniques

Step 1: Prove an exponential version of $q(k+1)u(k) = 0$

Step 2: Bound unused service and take heavy-traffic limit

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Moment Generating Function

$$\mathbb{E}\left[e^{\theta \epsilon q}\right]$$

- $\theta \in \mathbb{R}$
- Two-sided Laplace transform of stationary distribution
- Must exist for θ in an interval around the origin

  ➡Must prove

Characteristic Function

$$\mathbb{E}\left[e^{i\theta \epsilon q}\right]$$

- $\theta \in \mathbb{R}, i = \sqrt{-1}$
- Fourier transform of stationary distribution
- Must use complex numbers

One-Sided Laplace Transform

$$\mathbb{E}\left[e^{\theta \epsilon q}\right]$$

- $\theta < 0$
- Always exists because $q \geq 0$
- Must show existence of moments

Outline

Part 1: The Model	• Important parts of a queueing model • Supermarket-checkout model • Some routing algorithms
Part 2: How to Solve?	• Stability • Heavy-Traffic: The CRP condition • Analysis of a single-server queue • The MGF method
Part 3: An Open Question	• Many-server heavy-traffic regime • Comparison of routing algorithms open question
Conclusion and future work	

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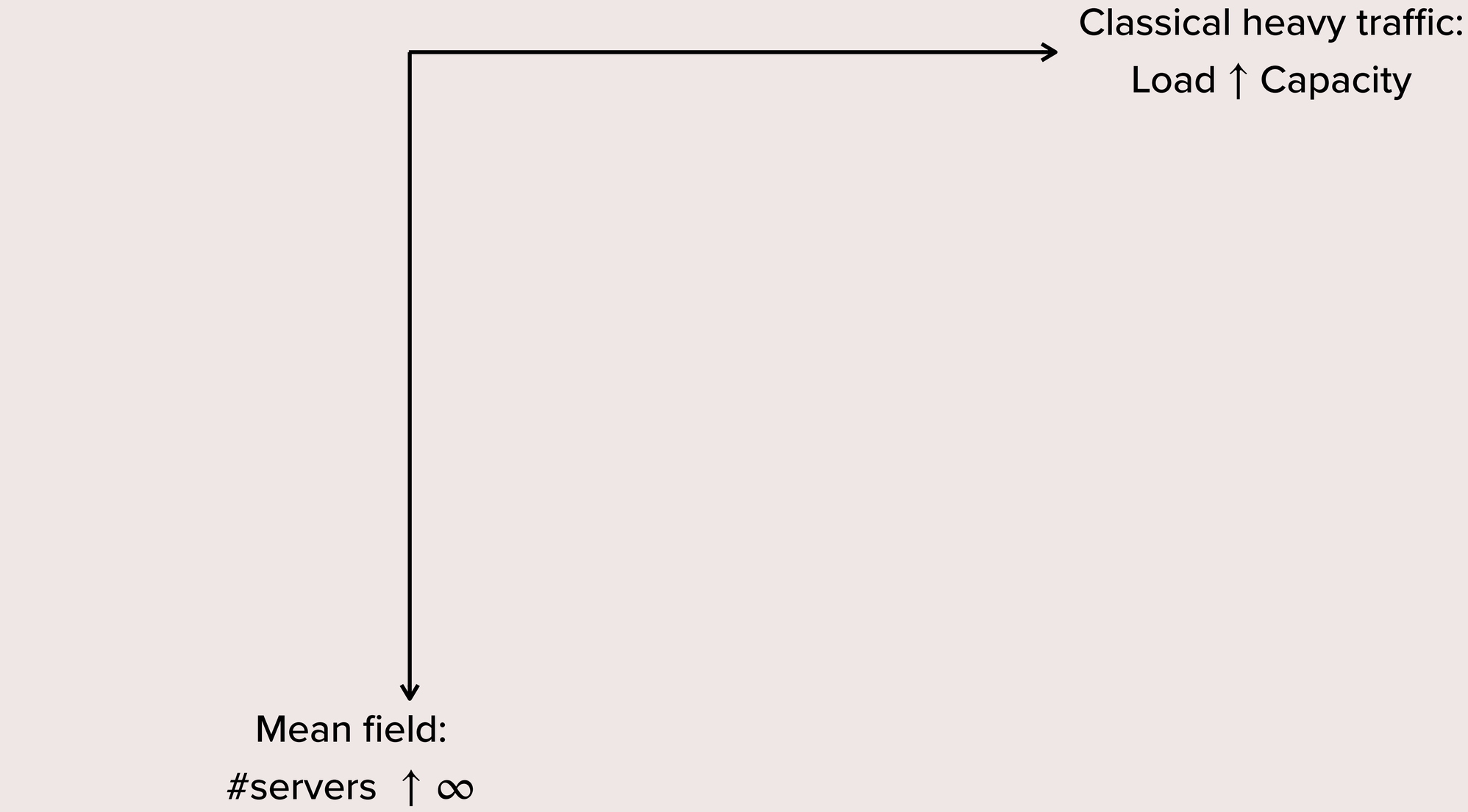
Many-Server Heavy-Traffic Regime



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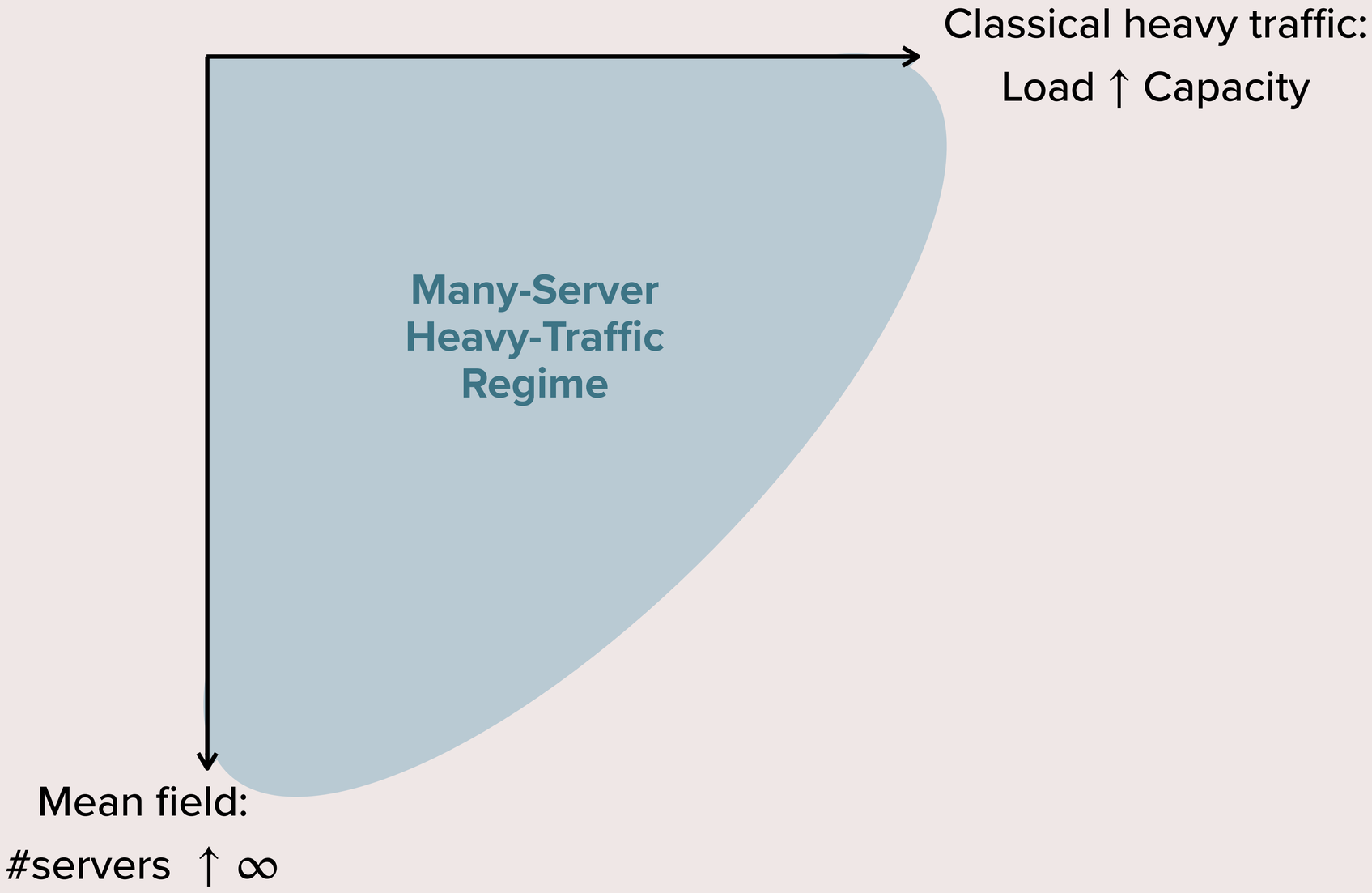
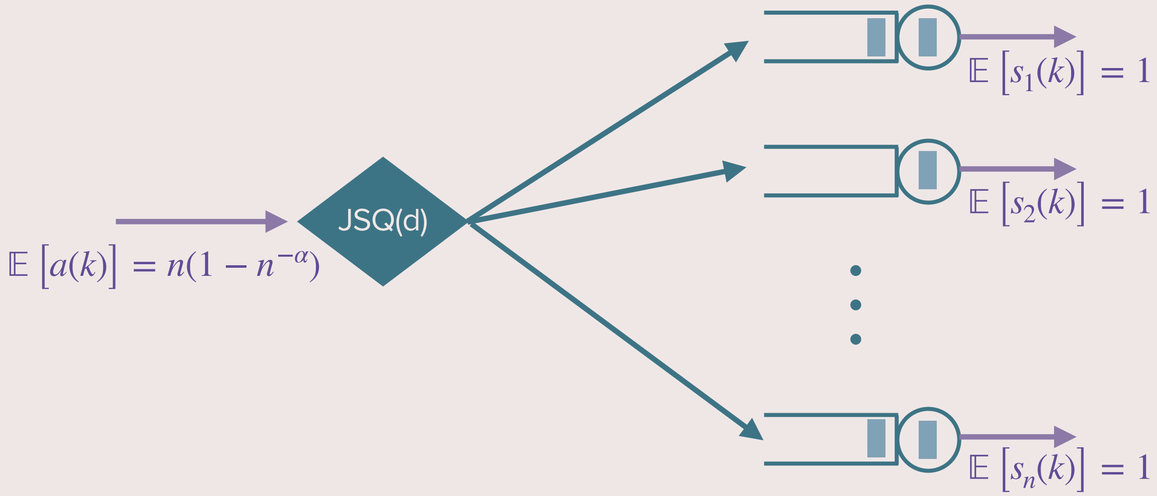


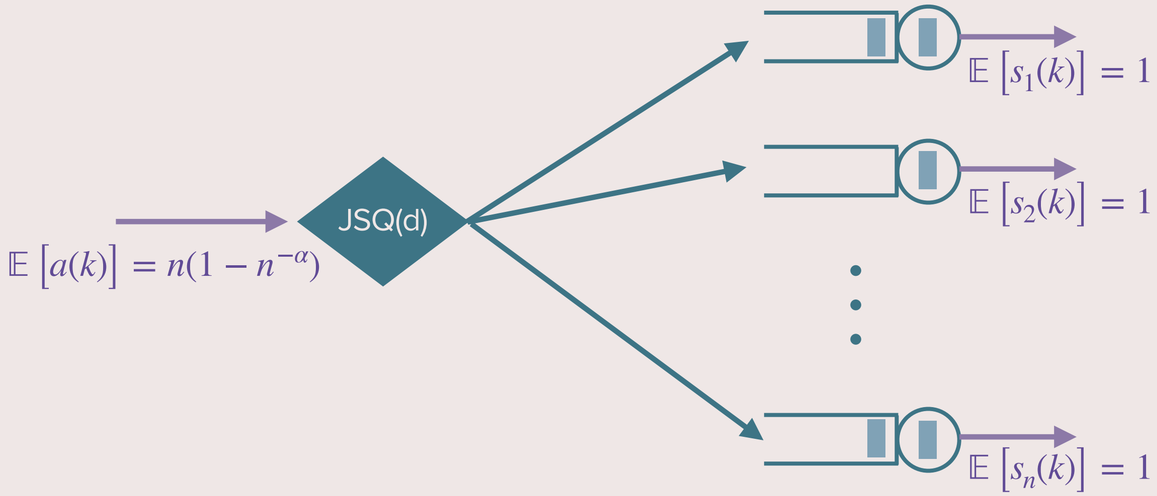
Many-Server Heavy-Traffic Parametrization



Many-Server Heavy-Traffic Parametrization

Service process:

- All servers are equal and independent of each other
- $\mathbb{E}\left[s_i(1)\right] = 1$



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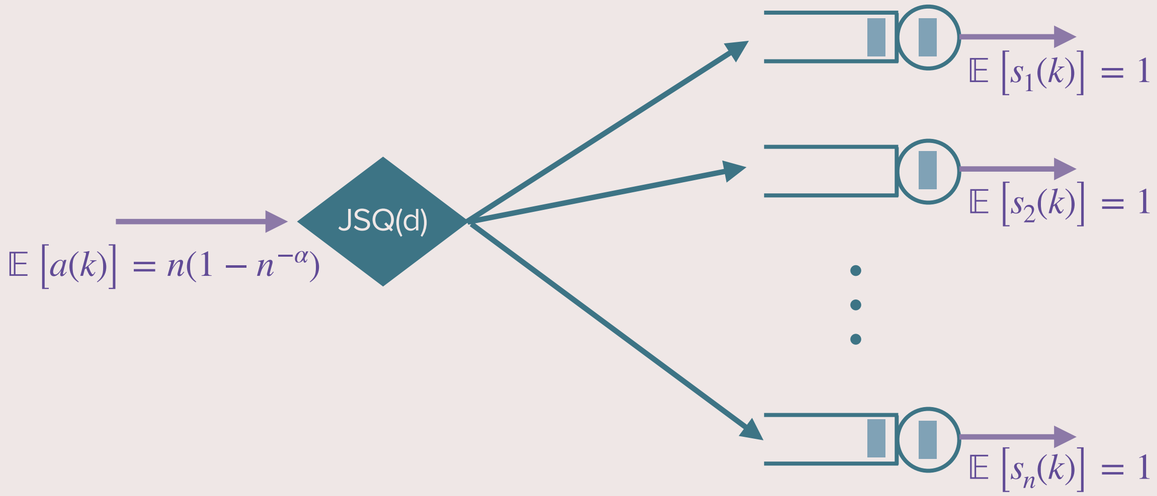

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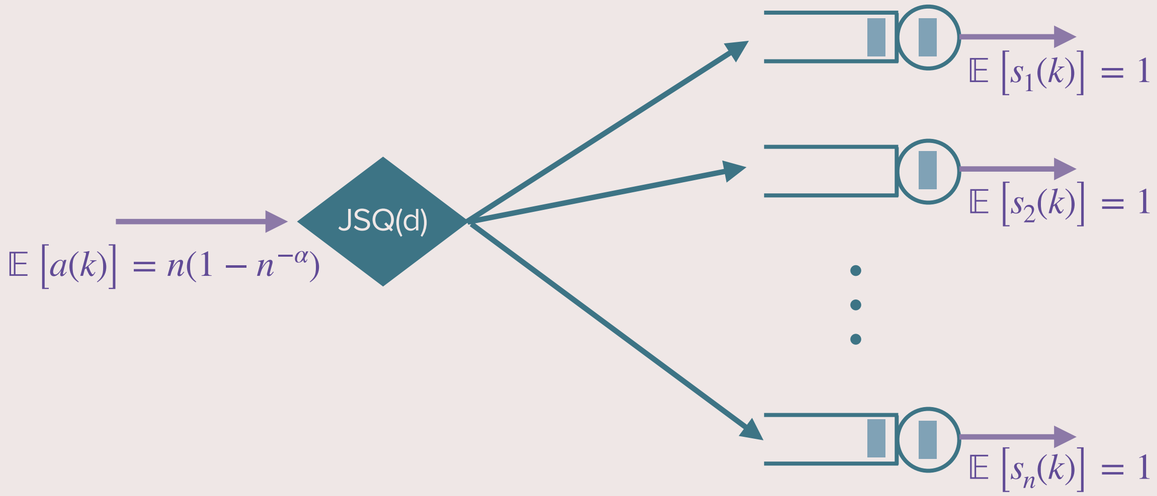
- Power-of-d choices



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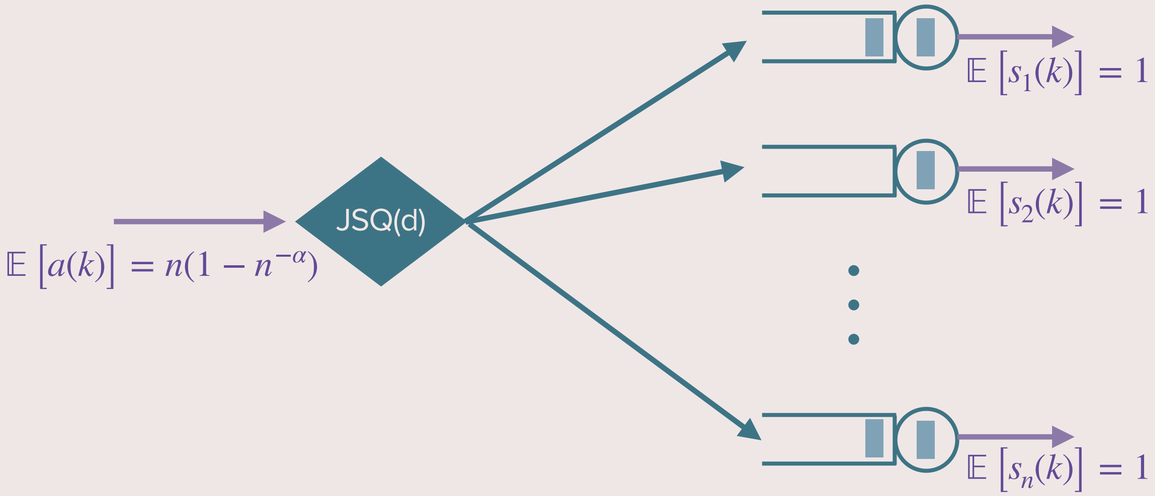

Many-server heavy-traffic limit:

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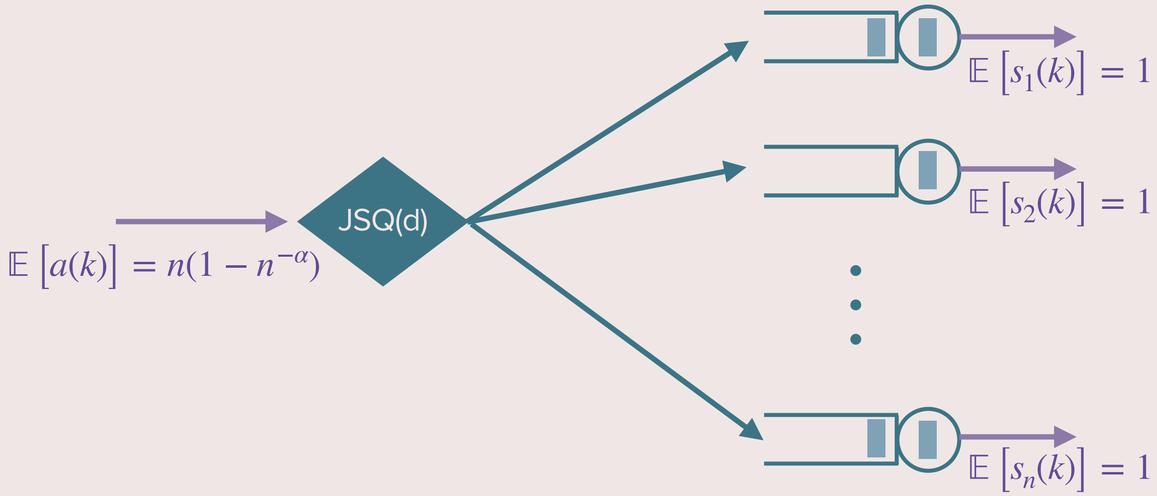

Many-server heavy-traffic limit:

- Take $n \uparrow \infty$

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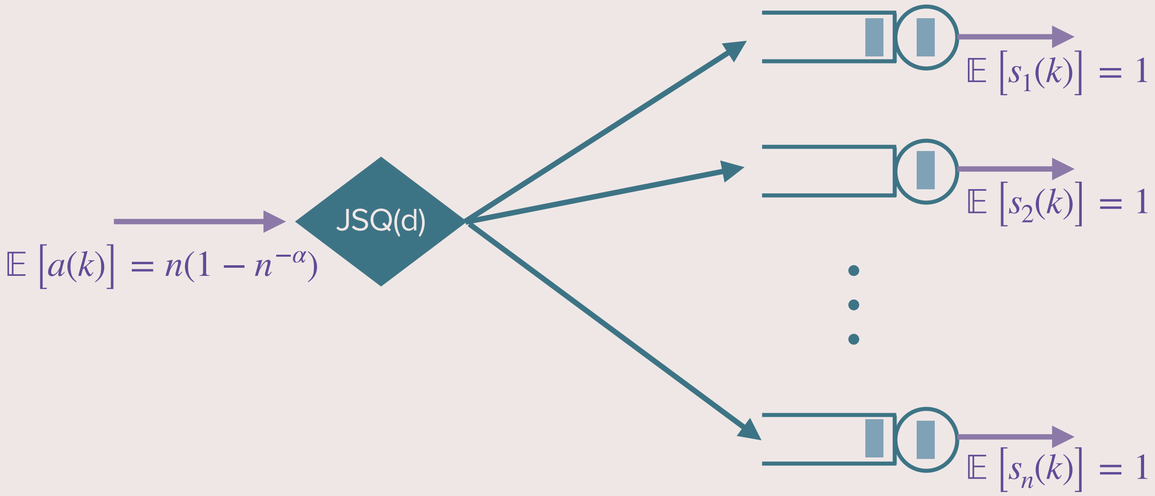


Many-server heavy-traffic limit:

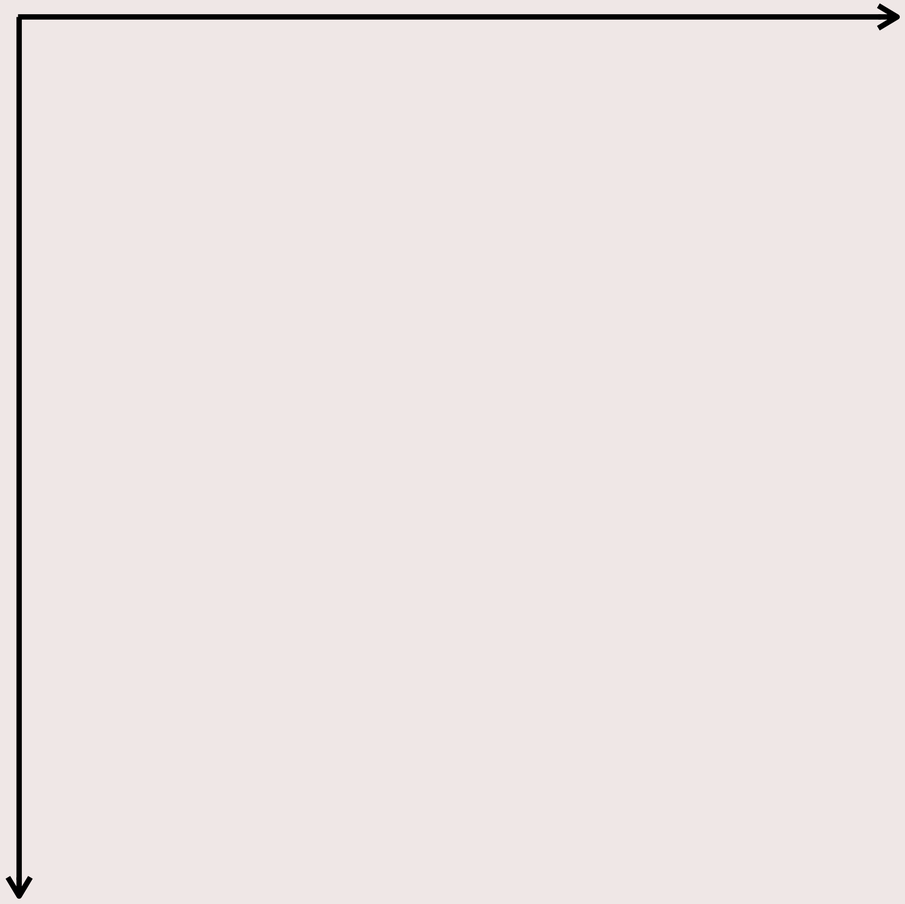
- Take $n \uparrow \infty$
- Then,
  - # servers $\uparrow \infty$
  - Load $\uparrow$ Max capacity

Many-Server Heavy-Traffic Regime

- Total service rate = n
- Arrival rate *per server* = $1 - n^{-\alpha}$

Classical heavy traffic:

"$\alpha \uparrow \infty$"



Mean field:

"$\alpha \downarrow 0$"

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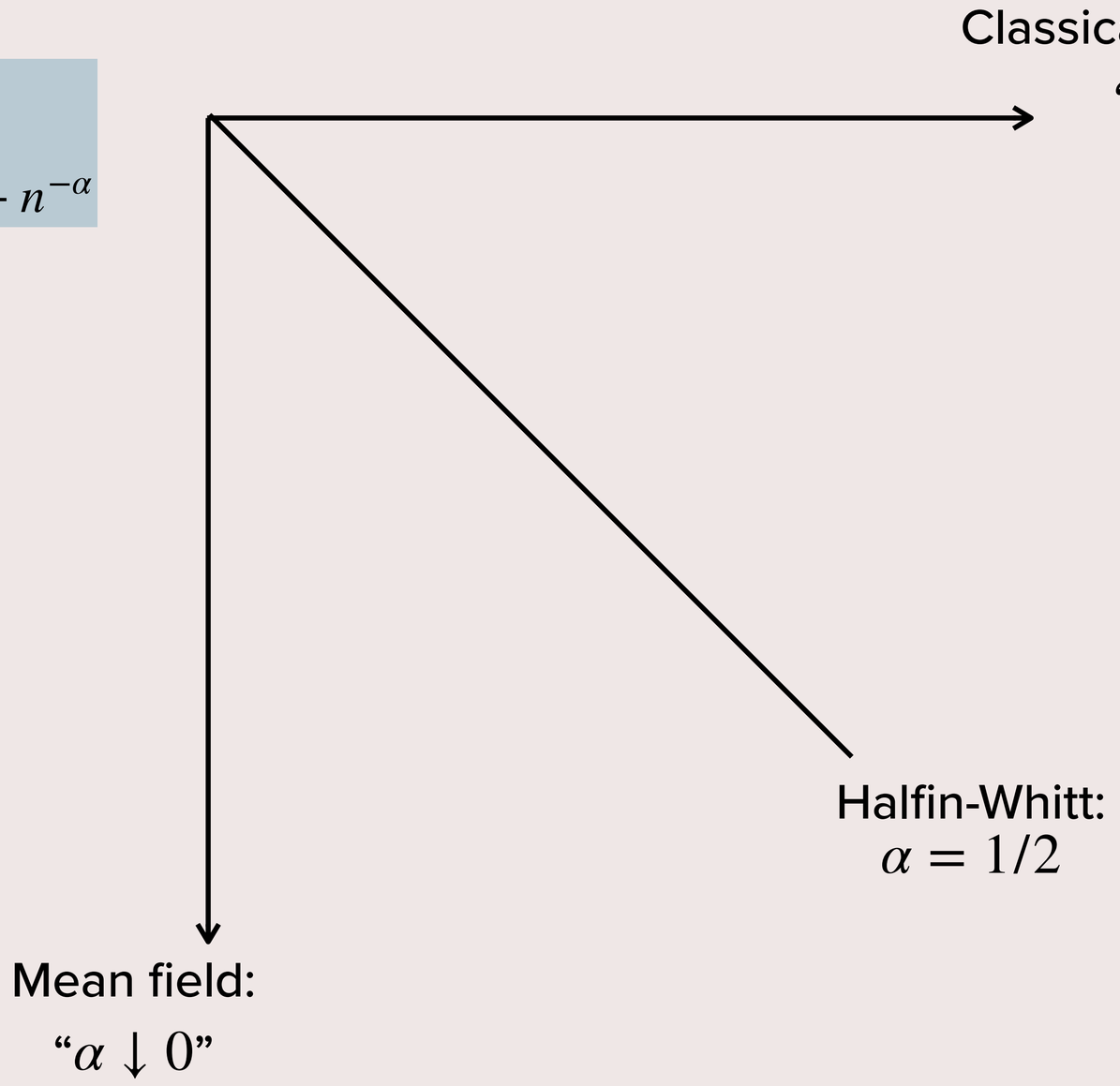

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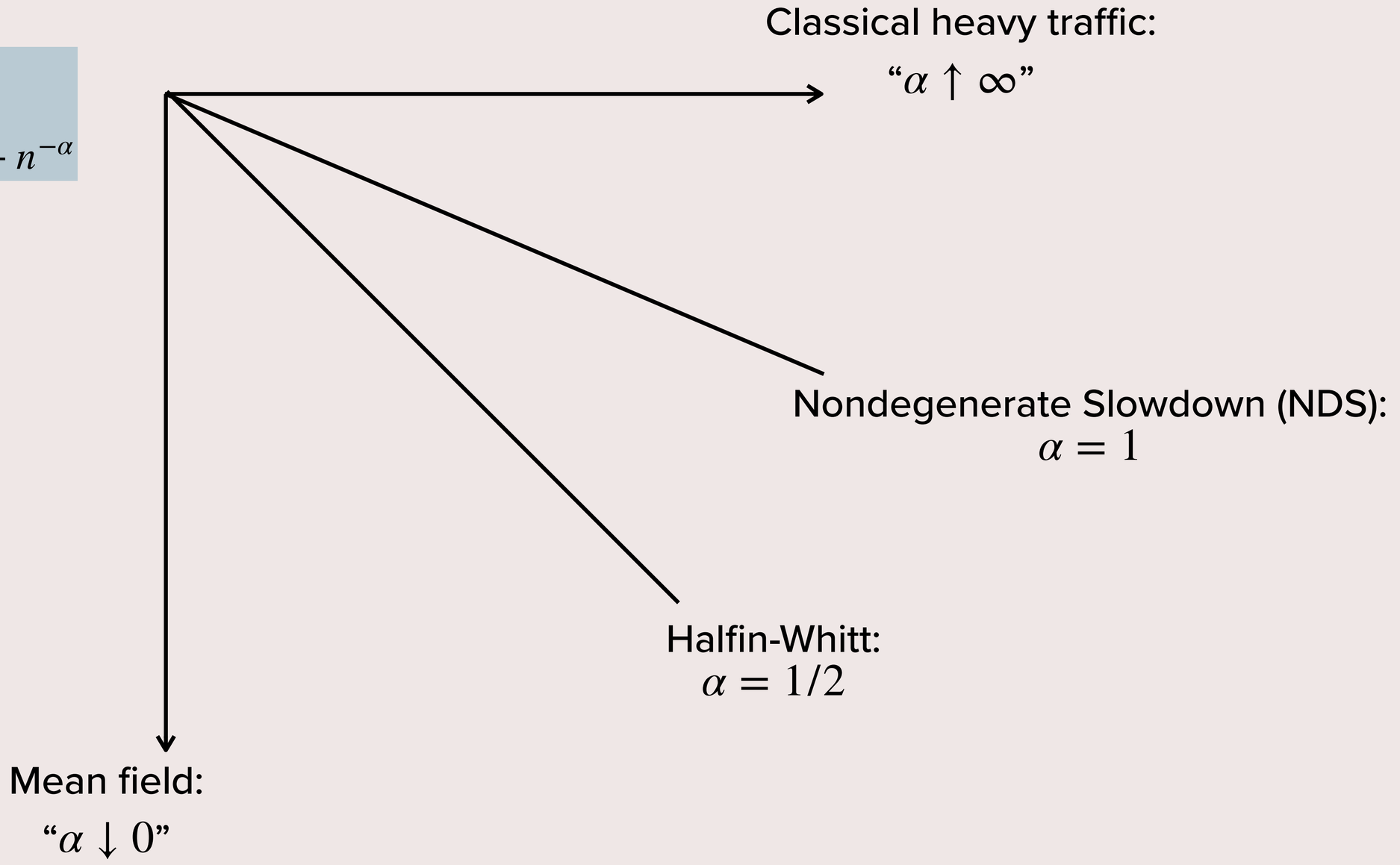

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$\alpha = 1$

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$\alpha = 1/2$

Halfin and Whitt (1981); Eschenfeldt and Gamarnik (2018); Banerjee and Mukherjee (2019,2020); Braverman (2020)...

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Value of $\alpha > 1$ to observe classical heavy-traffic behavior?

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Value of $\alpha > 1$ to observe classical heavy-traffic behavior?

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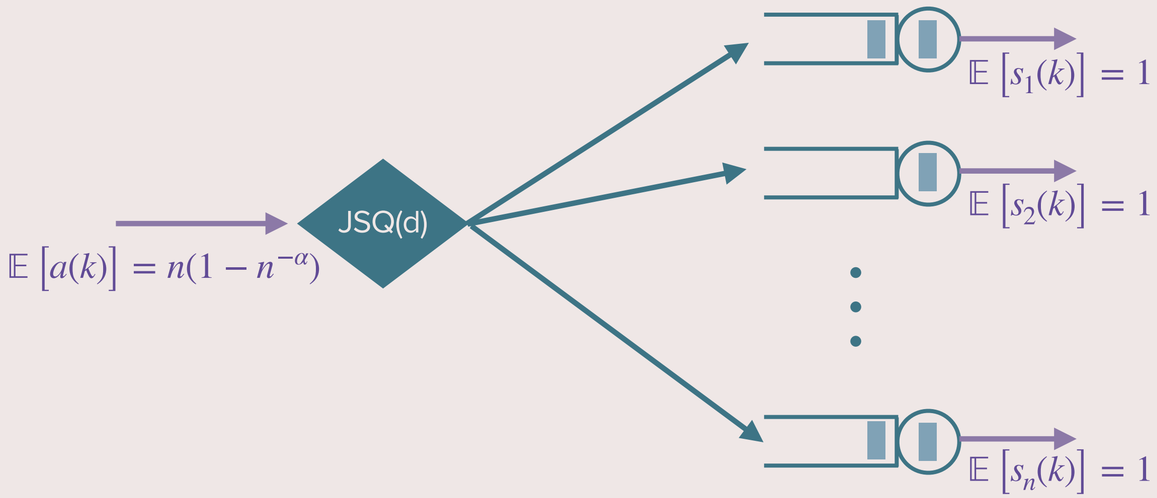
Before We Get to the Result...

Routing:

- Power-of-d choices
- In each time slot, route **all the arrivals to one server**



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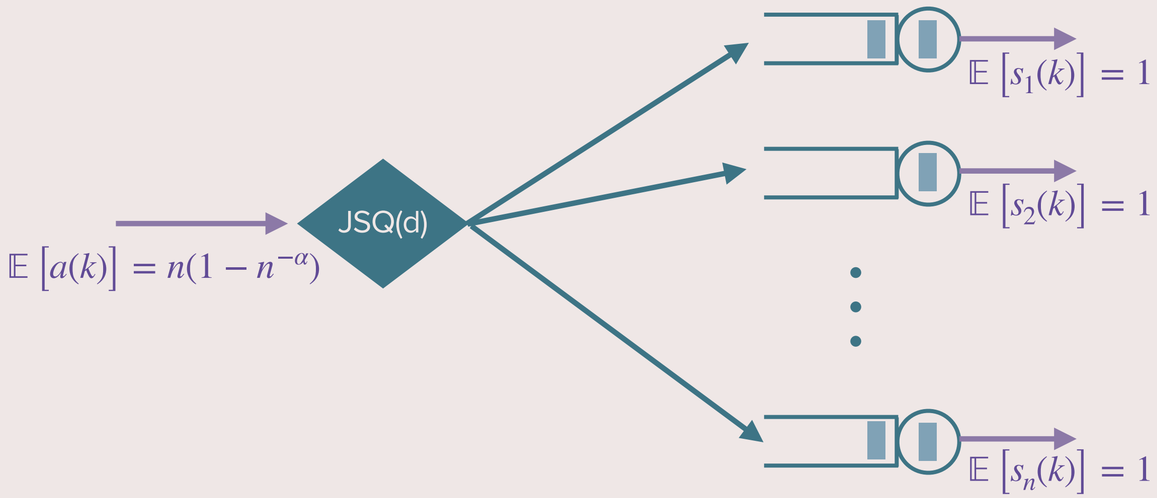

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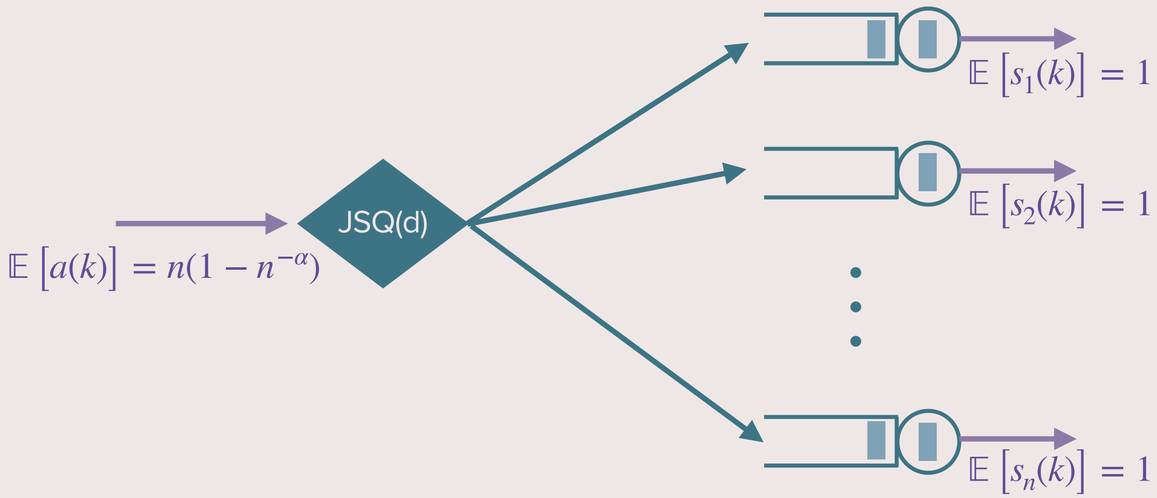

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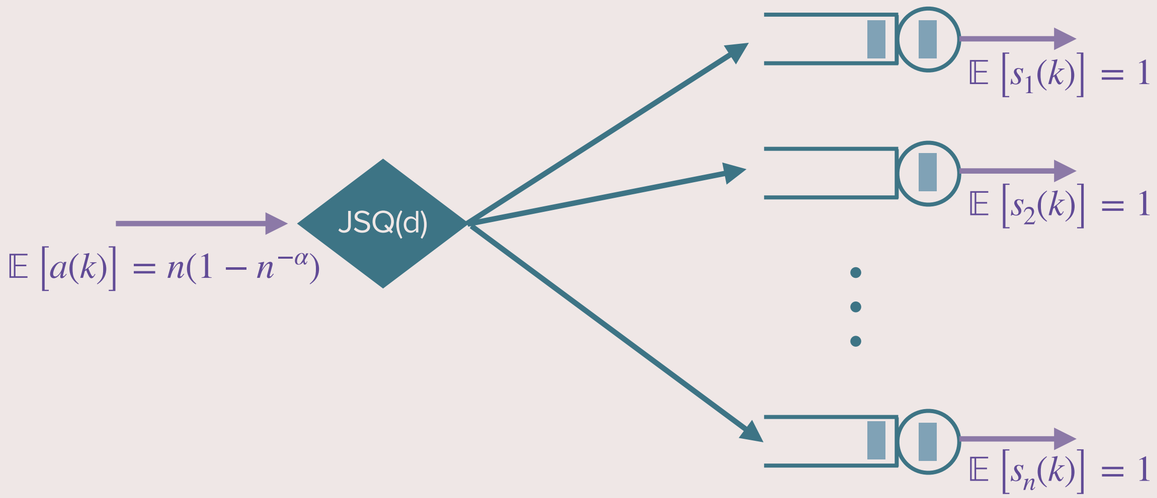

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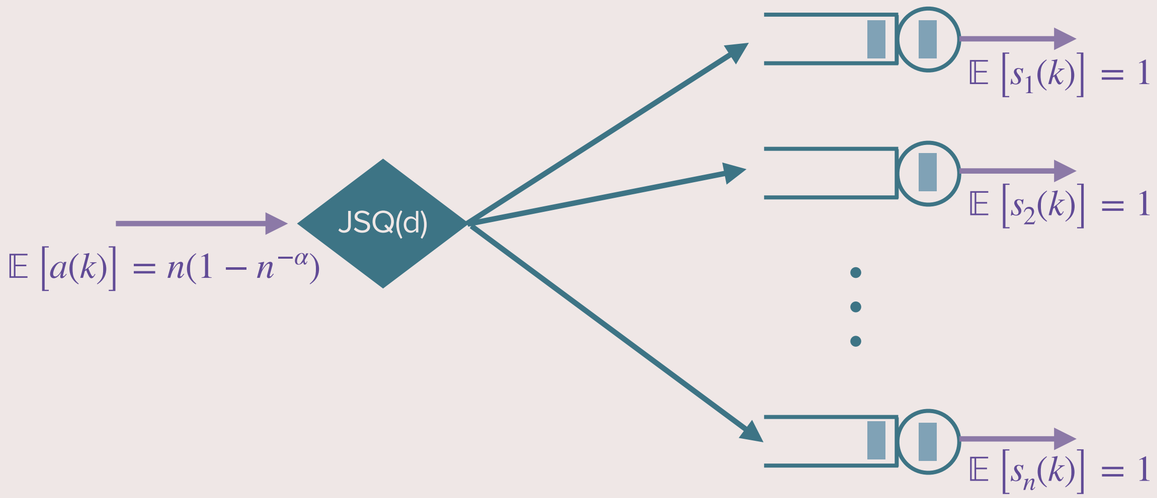

Warning!

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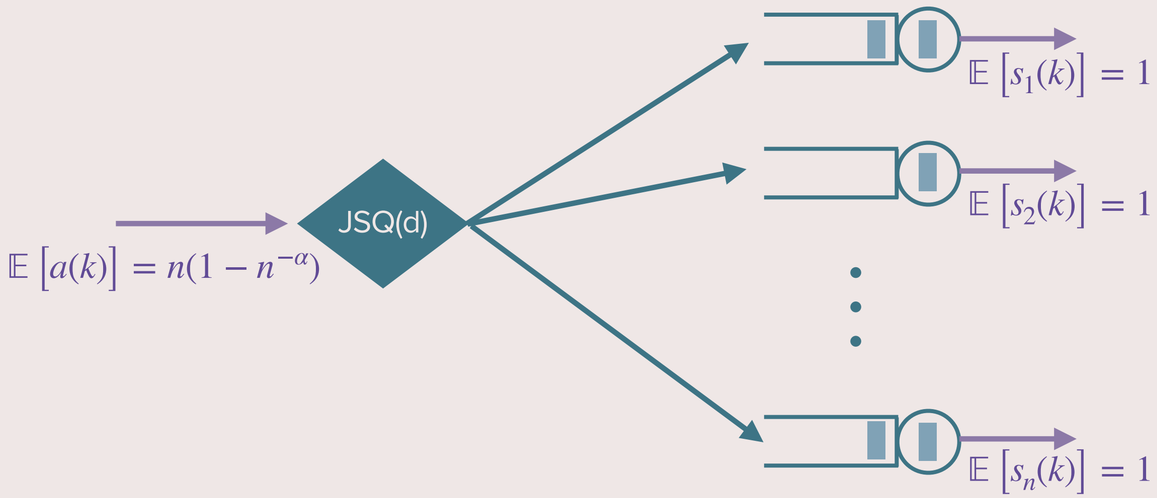
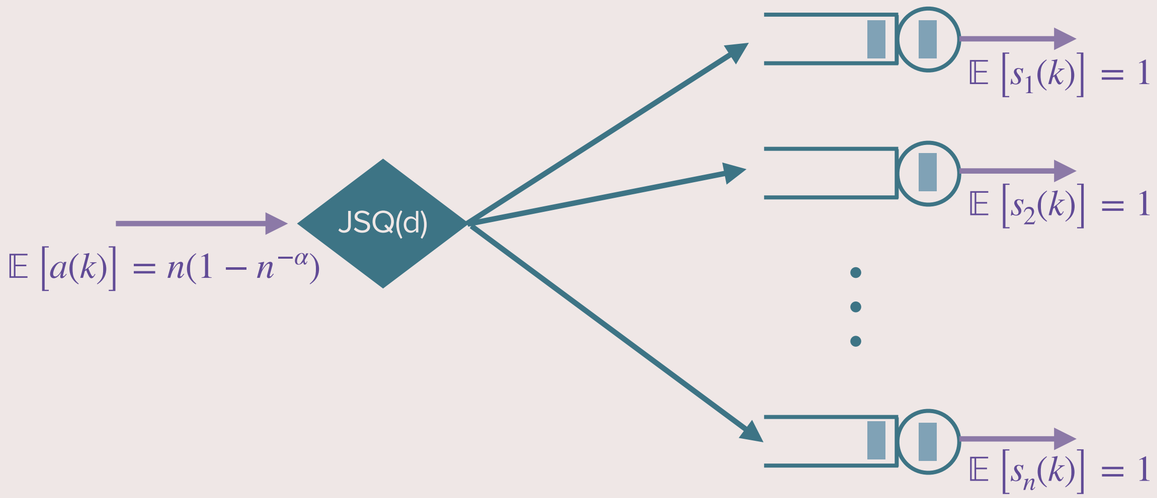


Warning!

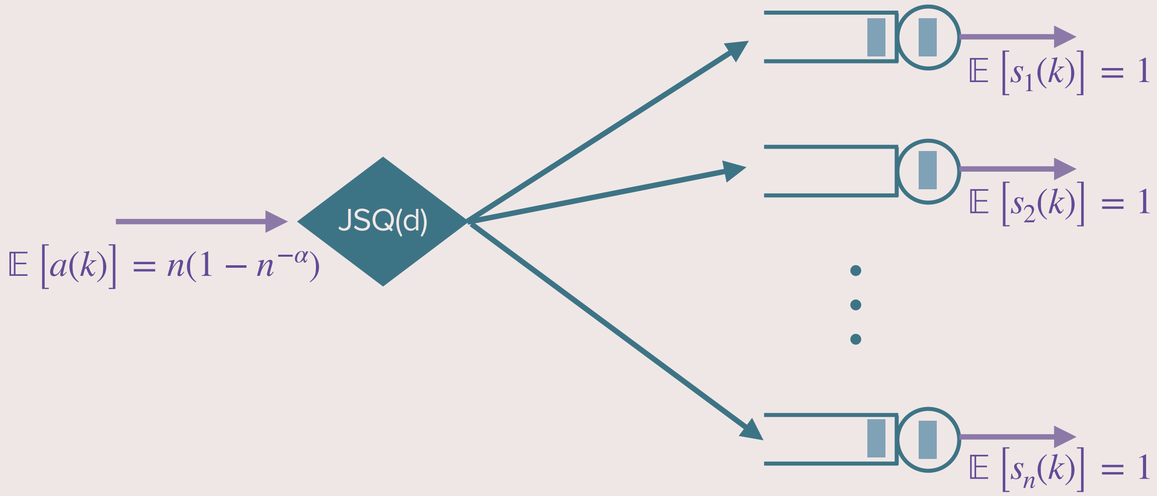
- We want classical heavy-traffic behavior
- We expect CRP condition
- In every time slot we route $O(n)$ jobs
- Then, convergence to CRP is weak

Supermarket Checkout Model in Continuous Time



Supermarket Checkout Model in Continuous Time

Model:



Supermarket Checkout Model in Continuous Time

Model:

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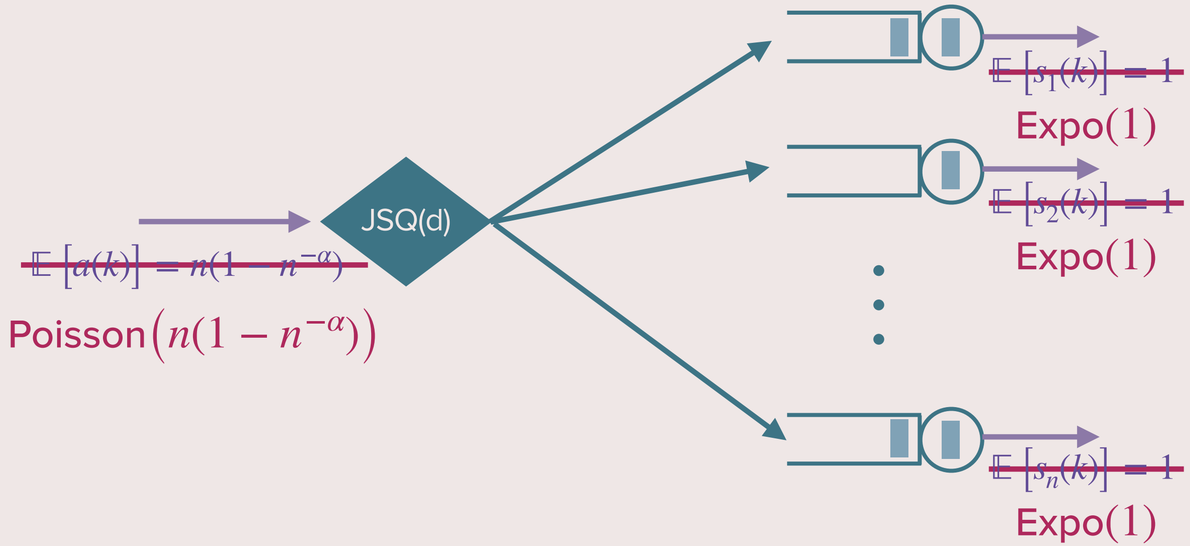

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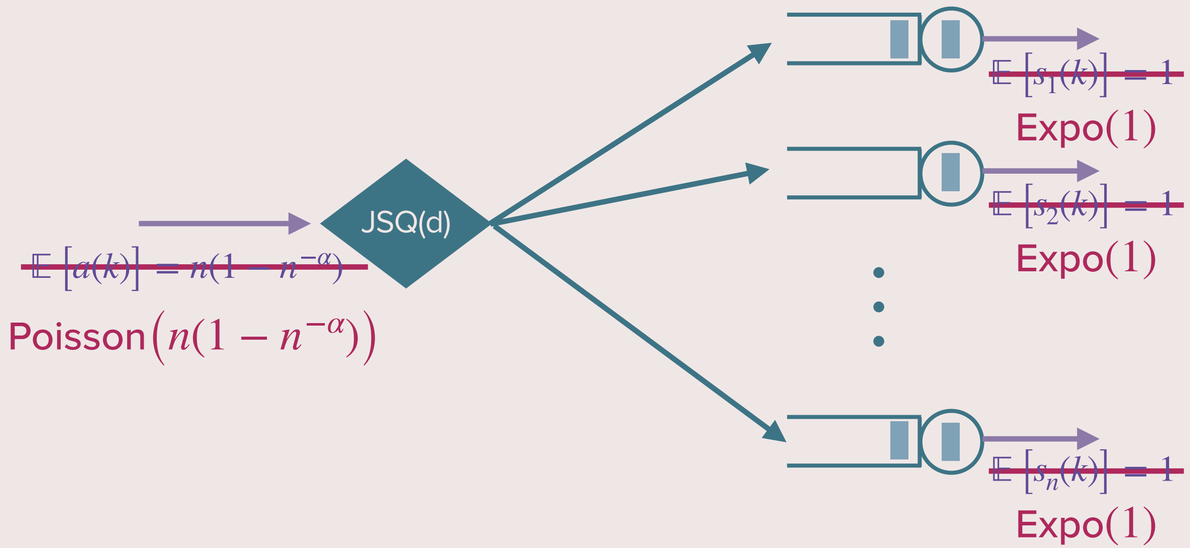
  - $d = cn^{\beta}, c > 0, \beta \geq 0$



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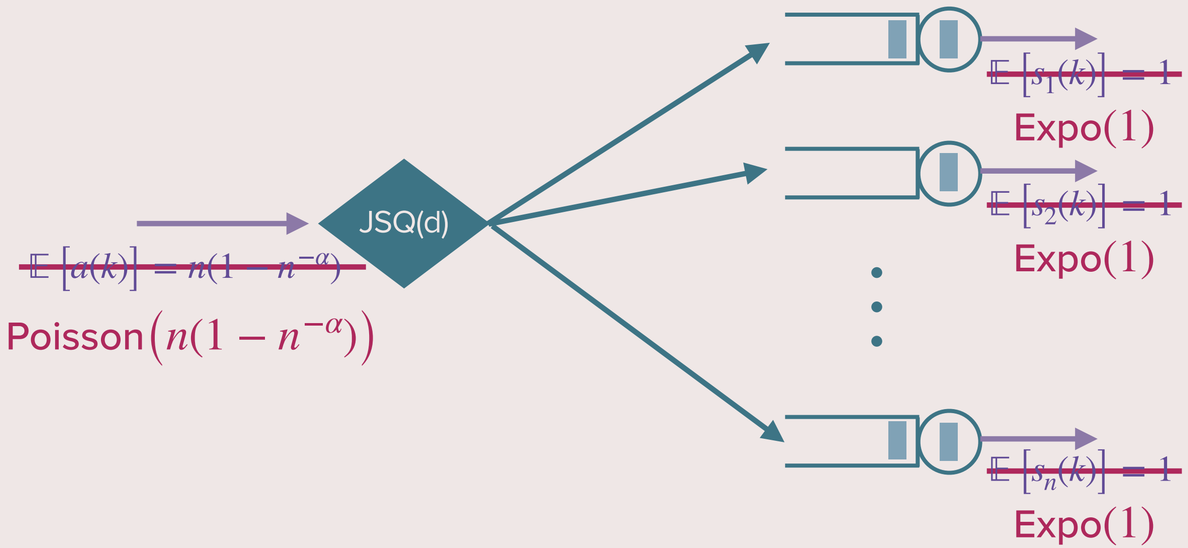

Theorem: [HL, Maguluri '21]

Consider a supermarket checkout system as described above. If $\alpha + \beta > 3$, then as $n \uparrow \infty$

$$n^{-\alpha} \sum_i q_i \Rightarrow Expo(1)$$

and $$\lim_{n\uparrow\infty} n^{-\alpha}\mathbb{E}\left[\sum_i q_i\right] = 1$$

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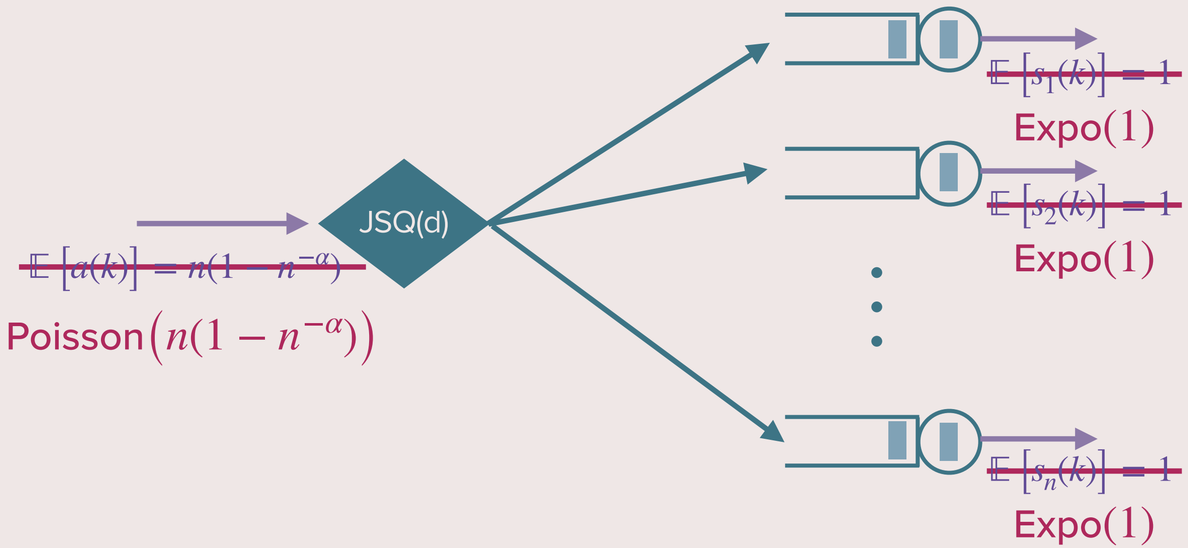

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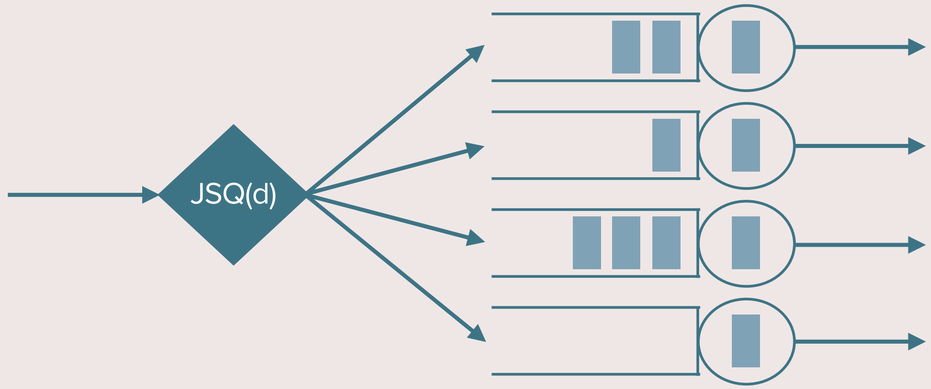
Proof:

1. Prove CRP
2. MGF method

Many-Server Heavy-Traffic: Which One is Better?



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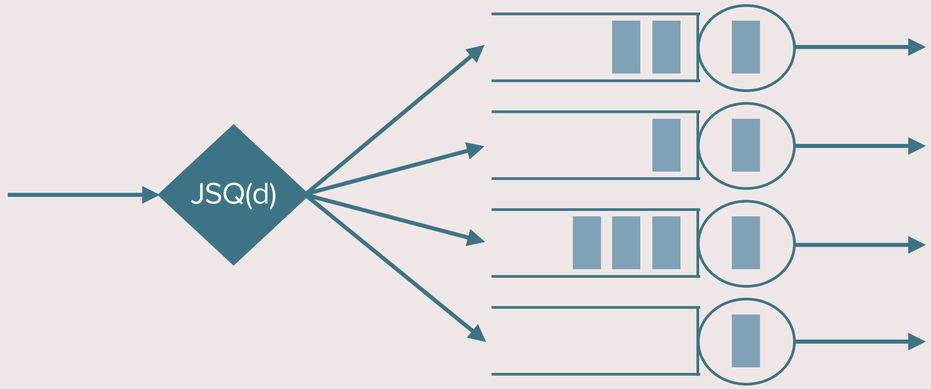

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JSQ

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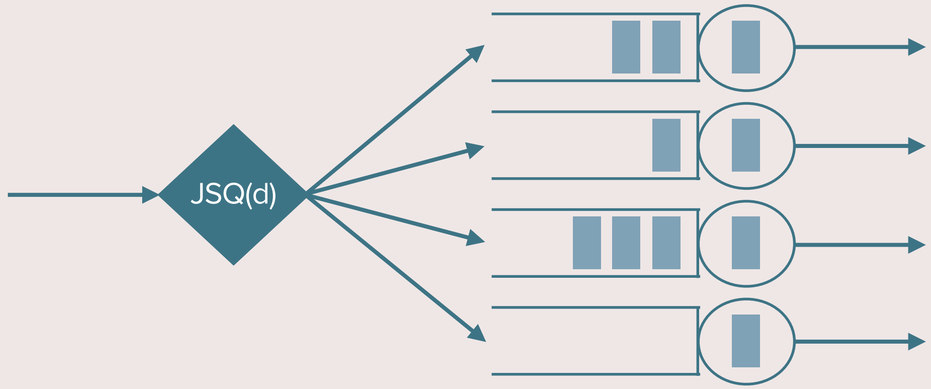

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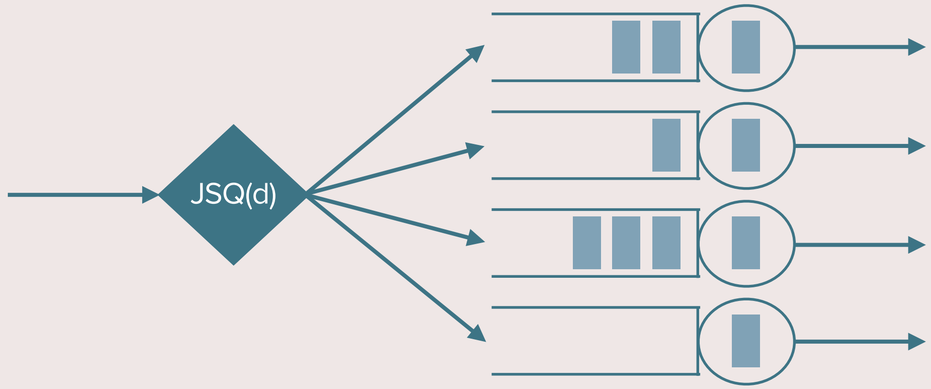

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JSQ(d)

$$d = cn^{\beta}$$
$$\alpha + \beta > 3$$

JSQ

$$\alpha > 2$$

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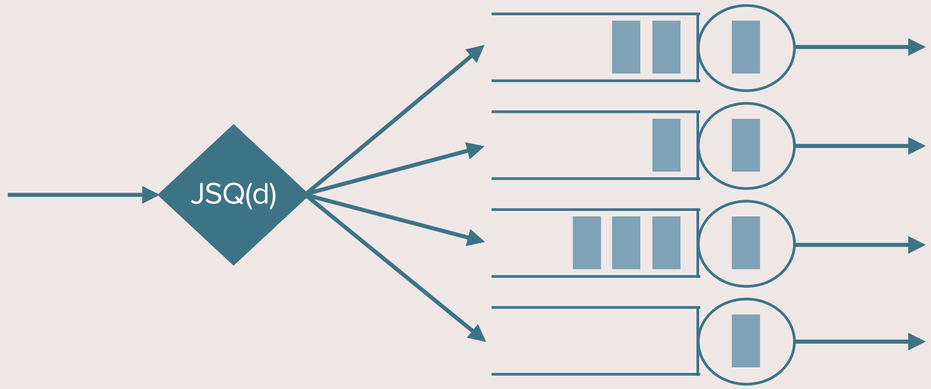

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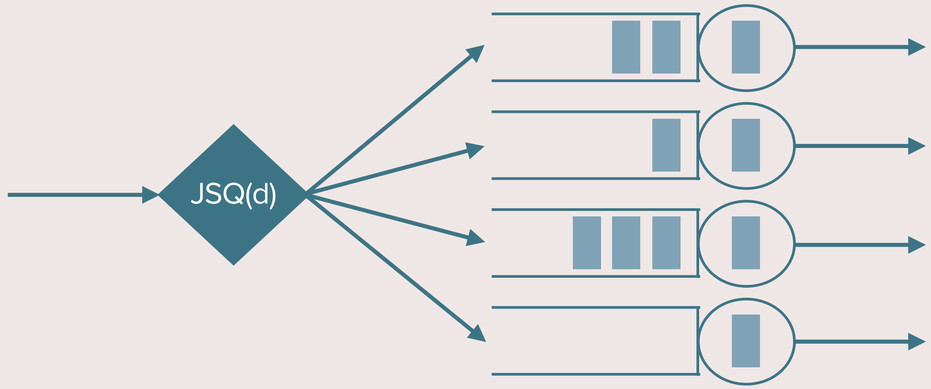

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Random $\leq$ **JSQ(d)** $<$ **JSQ**

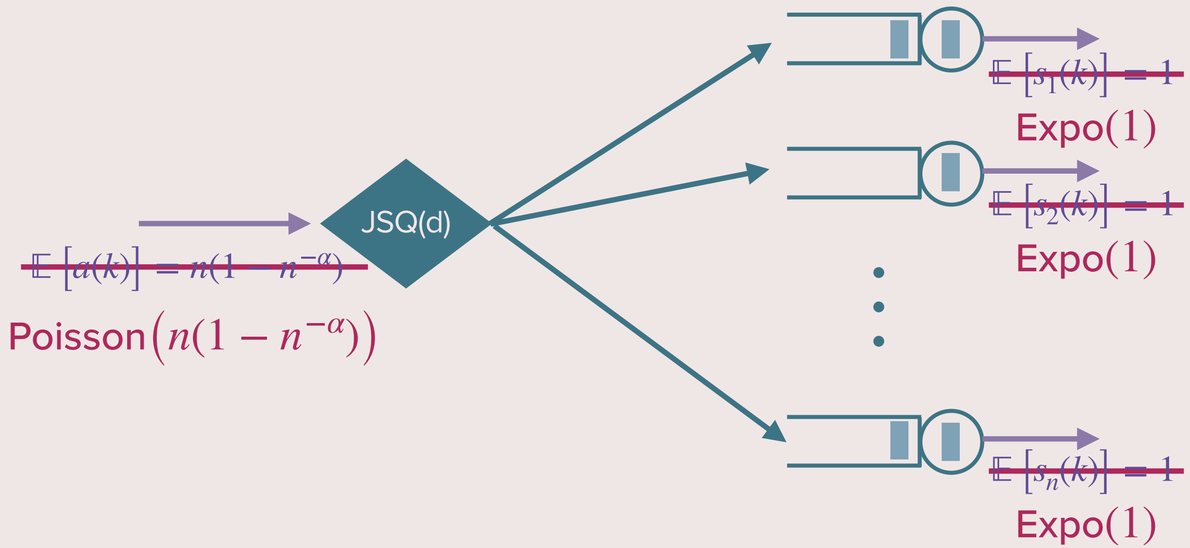
$\alpha > 3$

$d = cn^{\beta}$

$\alpha + \beta > 3$

$\alpha > 2$

Why can't we get $\alpha > 1$?



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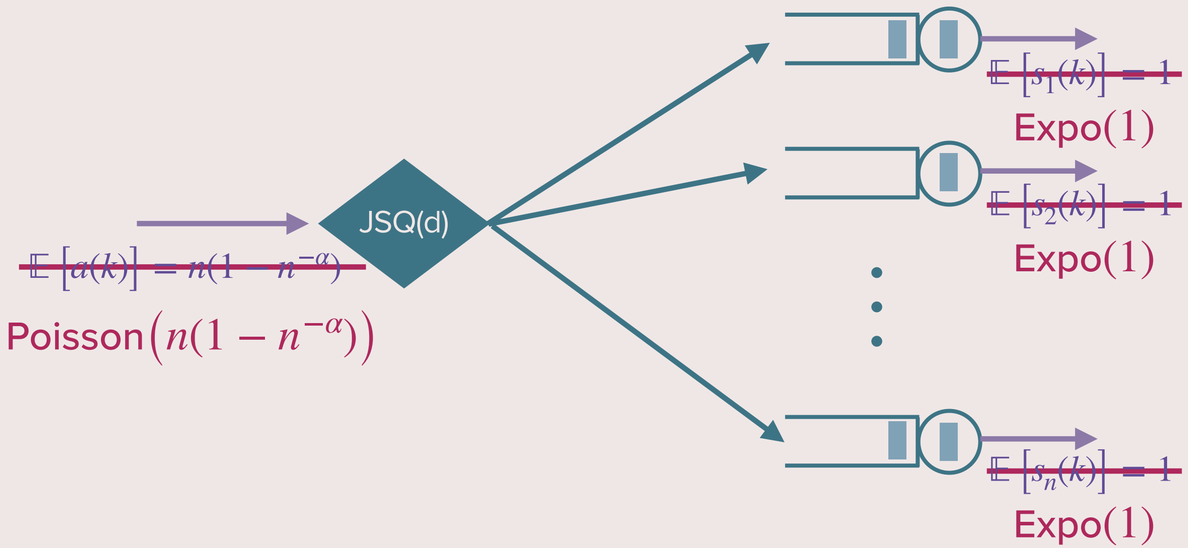

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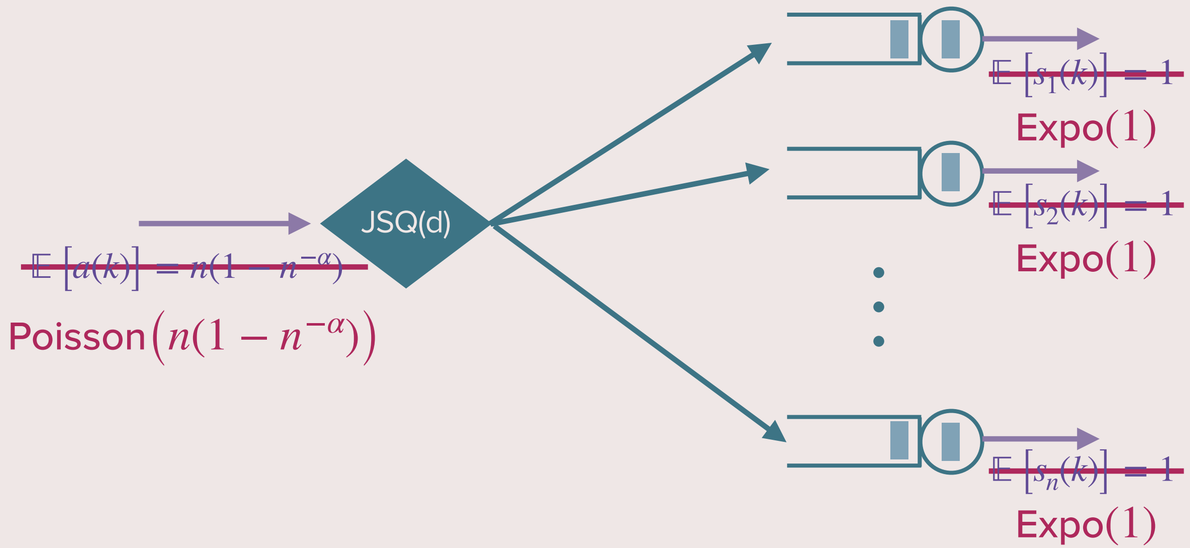

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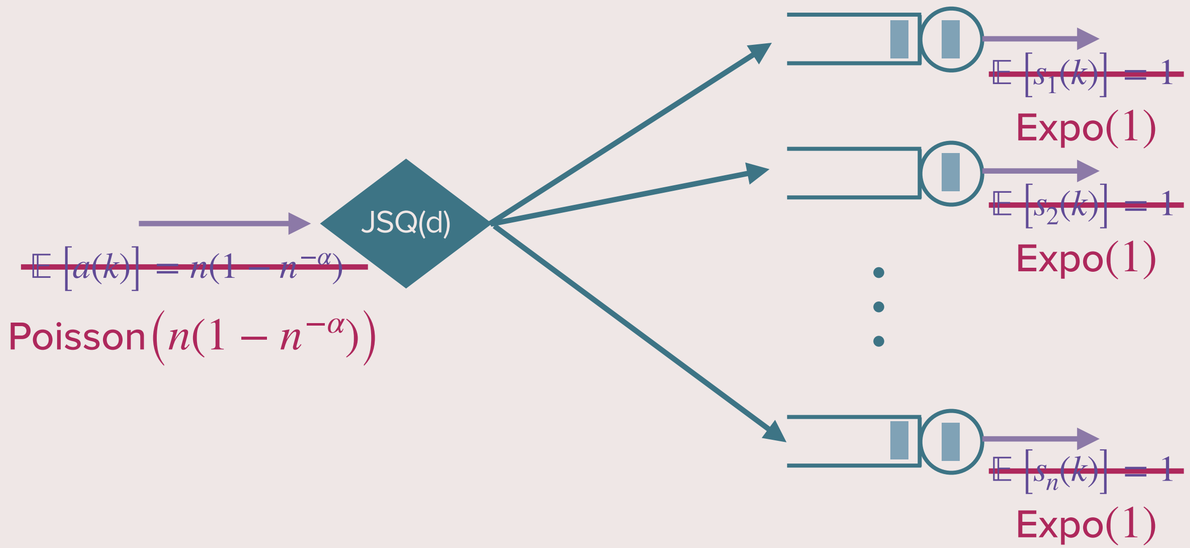

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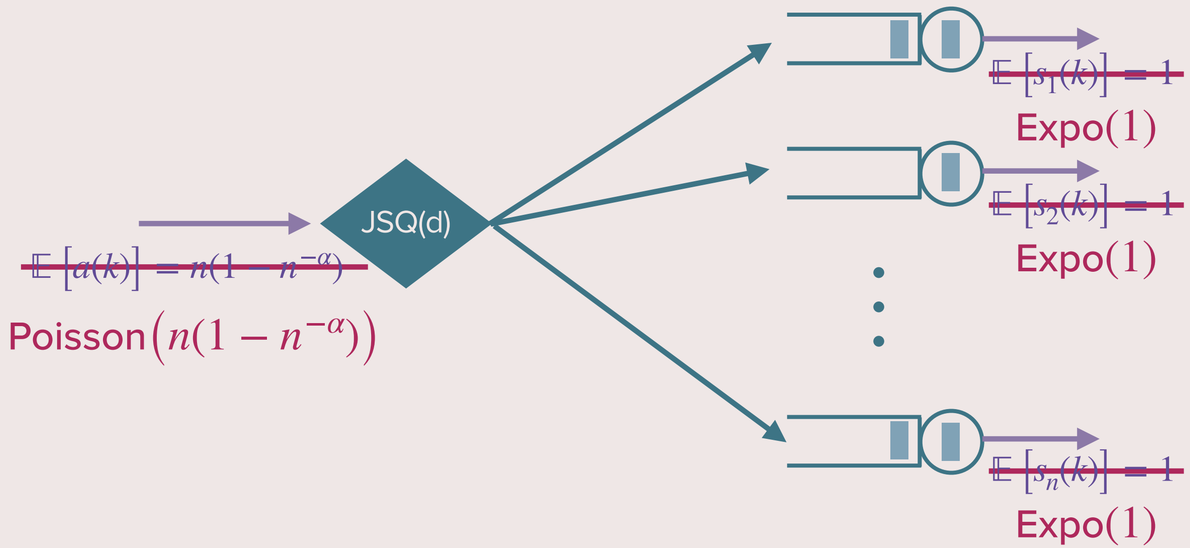

Back to discrete time:

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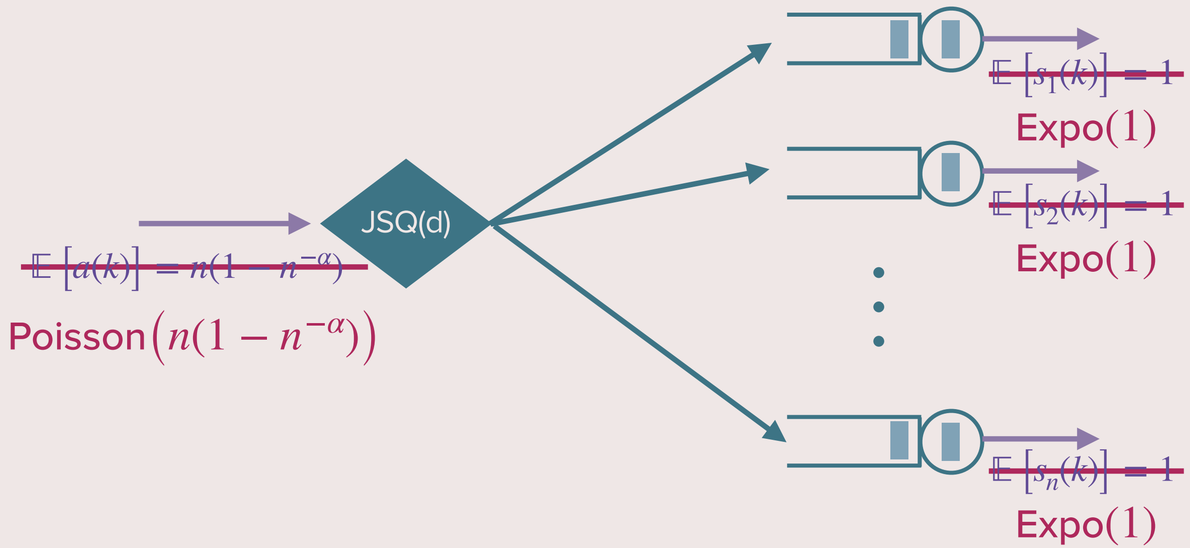

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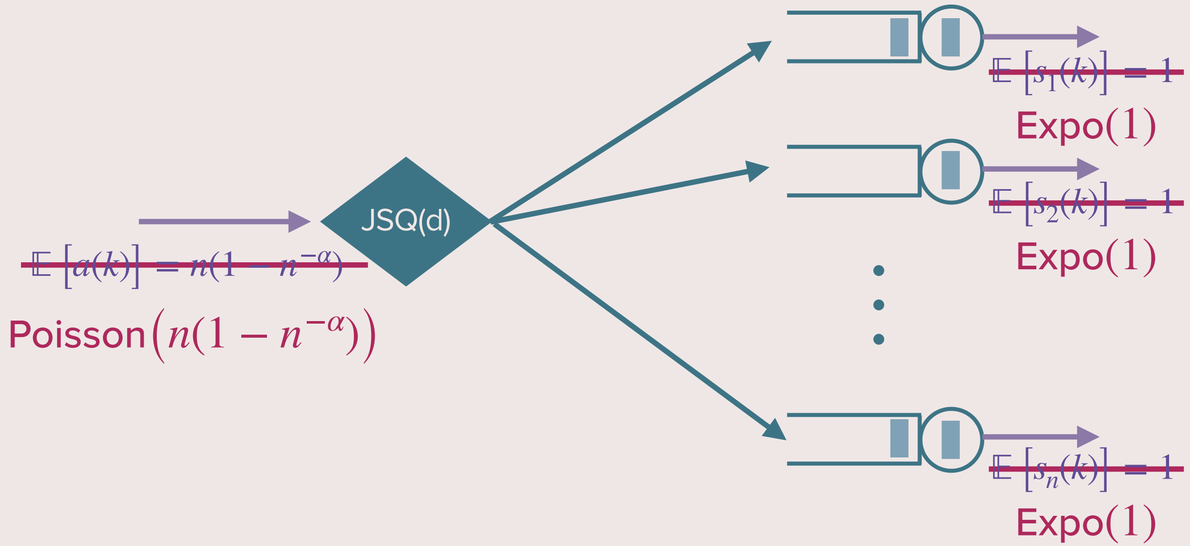

Back to discrete time:

- Current method: $\alpha > 11/2$
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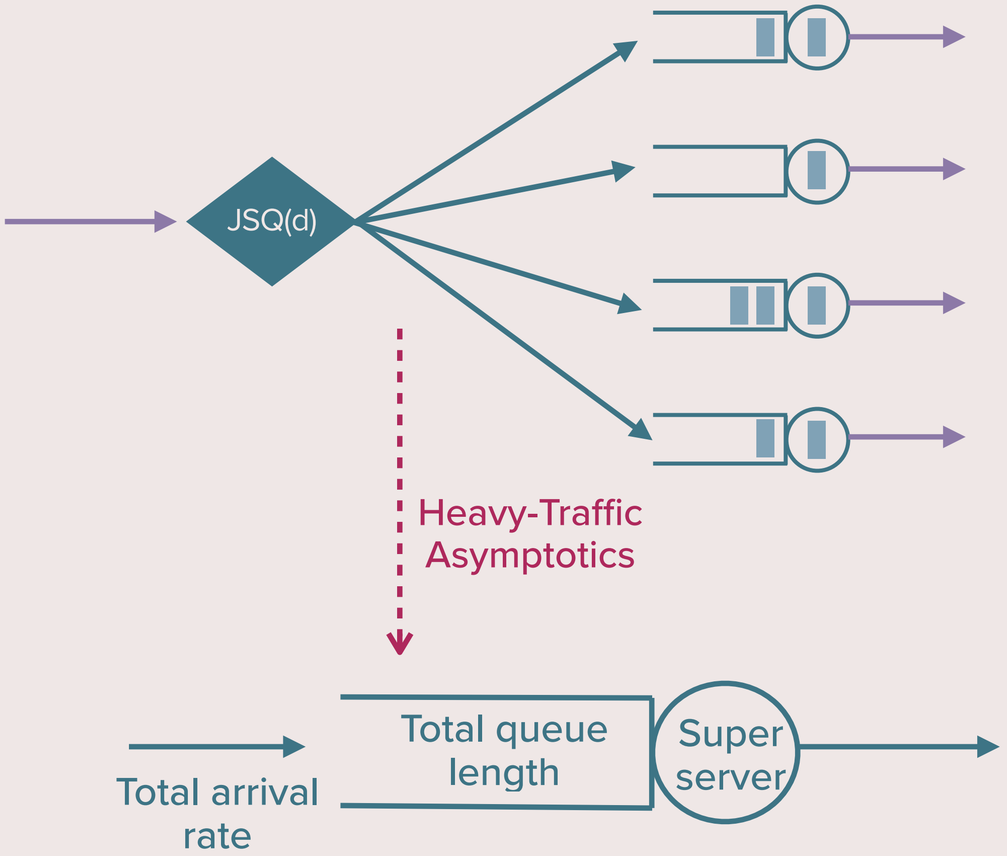
  - Can we reproduce continuous-time results?
  - Can we improve them?

Conclusion and Future Work



Conclusion and Future Work

Classical Heavy-Traffic Regime:

- CRP condition = Optimality
- MGF method: Compute the distribution
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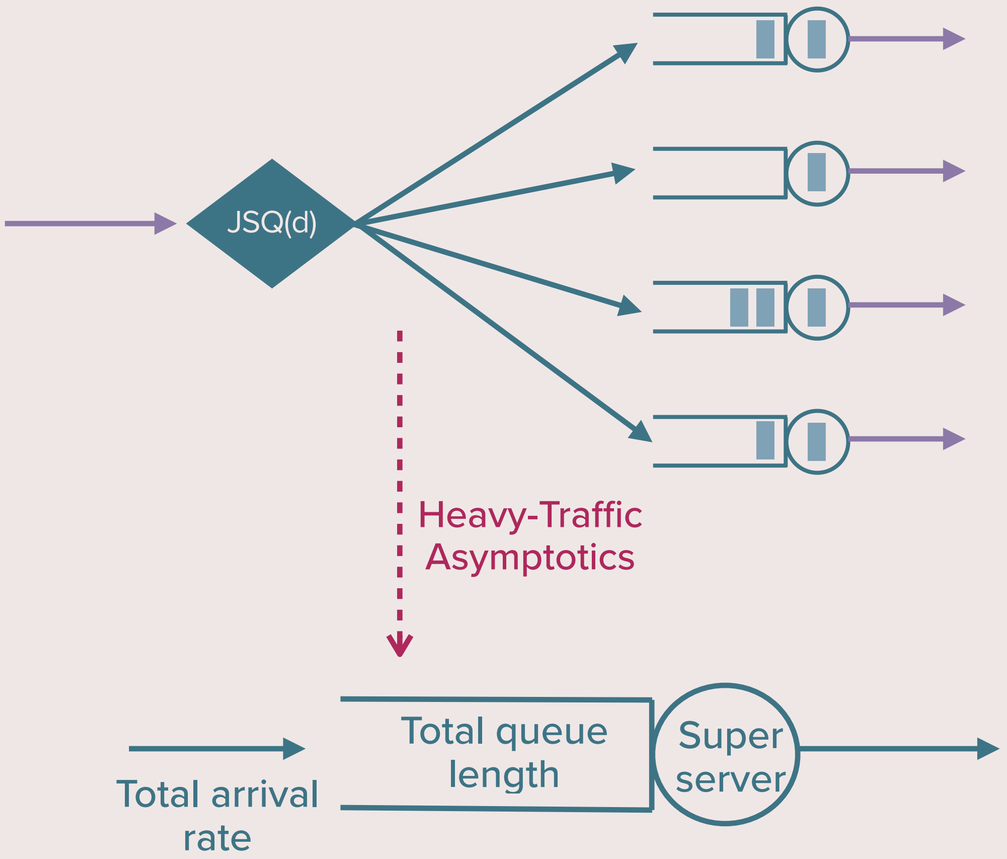
  - More than just heavy-traffic results
  - What to do in non-CRP systems?



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Classical Heavy-Traffic Regime:

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Many-server heavy-traffic regime:

- How fast should the load grow with respect to the number of servers to observe classical heavy traffic?
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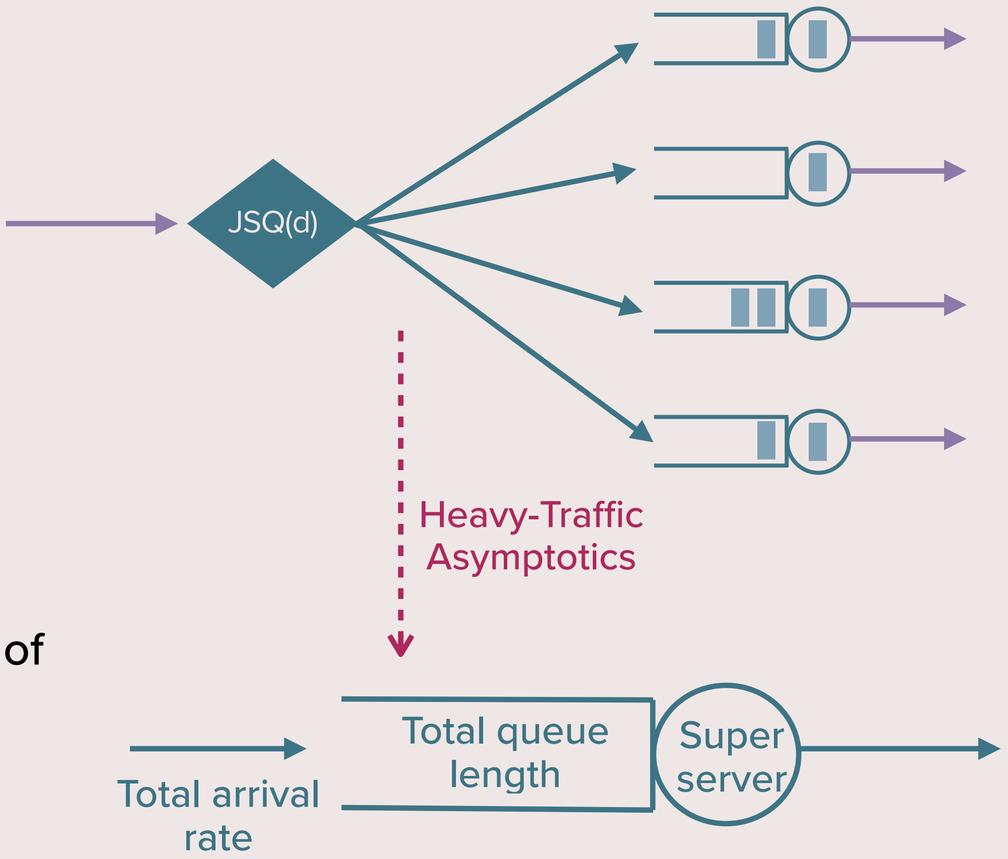
  - JSQ: $\alpha > 2$
  - What happens for $\alpha \in (1,2]$?
  - Stronger convergence to CRP? Another distribution?



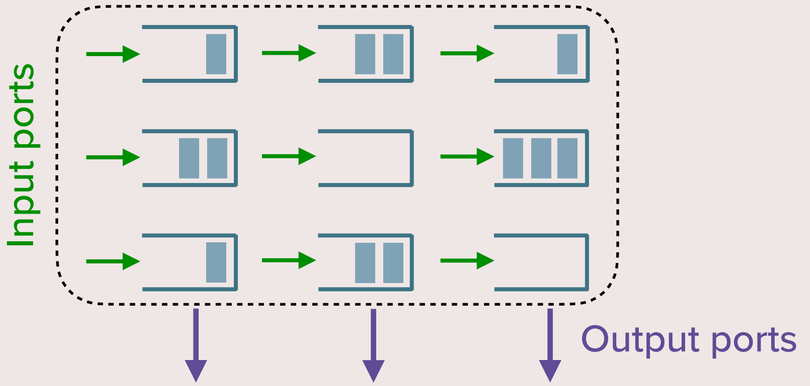
Minimizing Delay in Supermarket-Checkout Systems

Daniela Hurtado-Lange
William and Mary
Joint work with Siva Theja Maguluri

Mathematics Colloquium, April 1st, 2022

Contact information: dahurtadolange@wm.edu

The Input-Queued Switch Model



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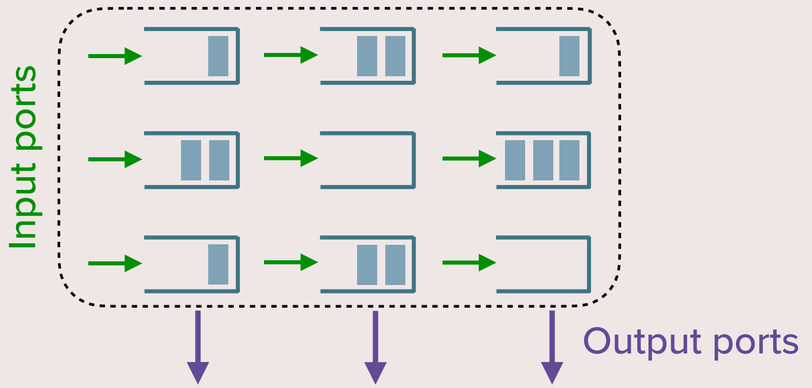

The Input-Queued Switch Model

- Discrete time model
- n input ports and n output ports
  - Jobs arrive at input ports, and go to the desired output port
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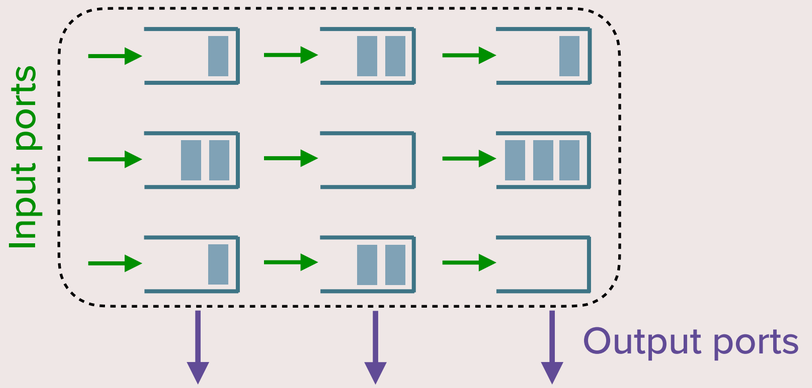

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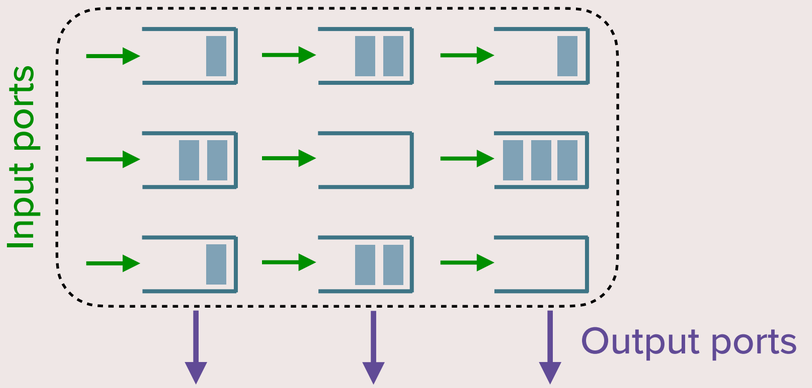

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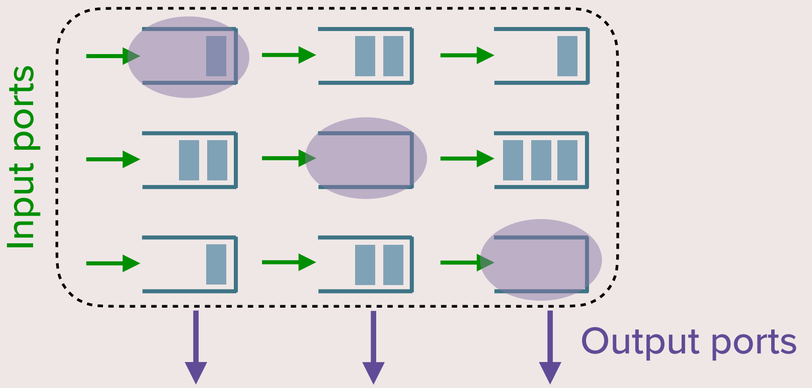

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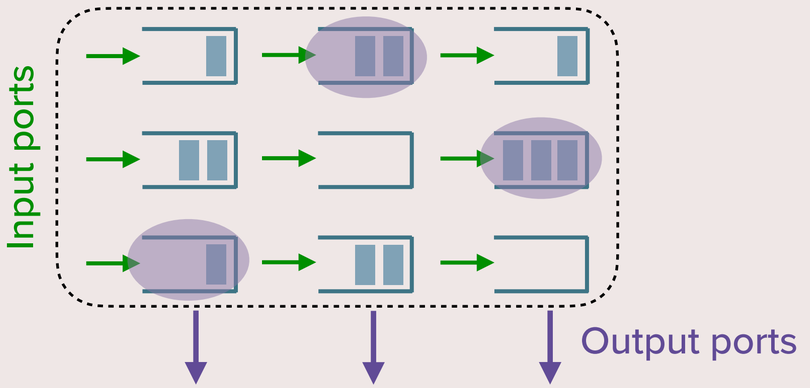

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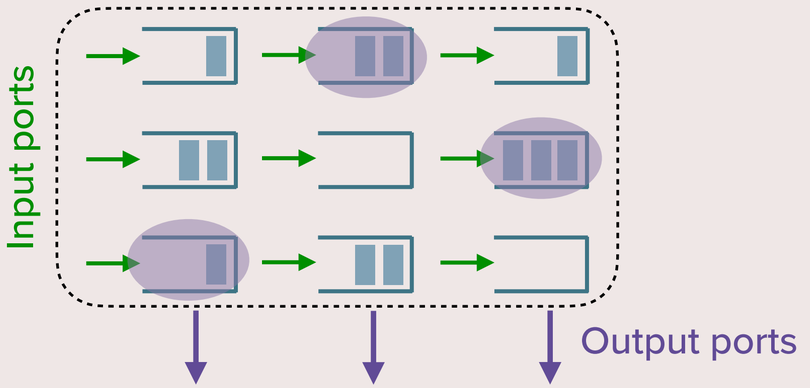
- **Scheduling:**
  - **Which queues to serve?** Largest total queue length

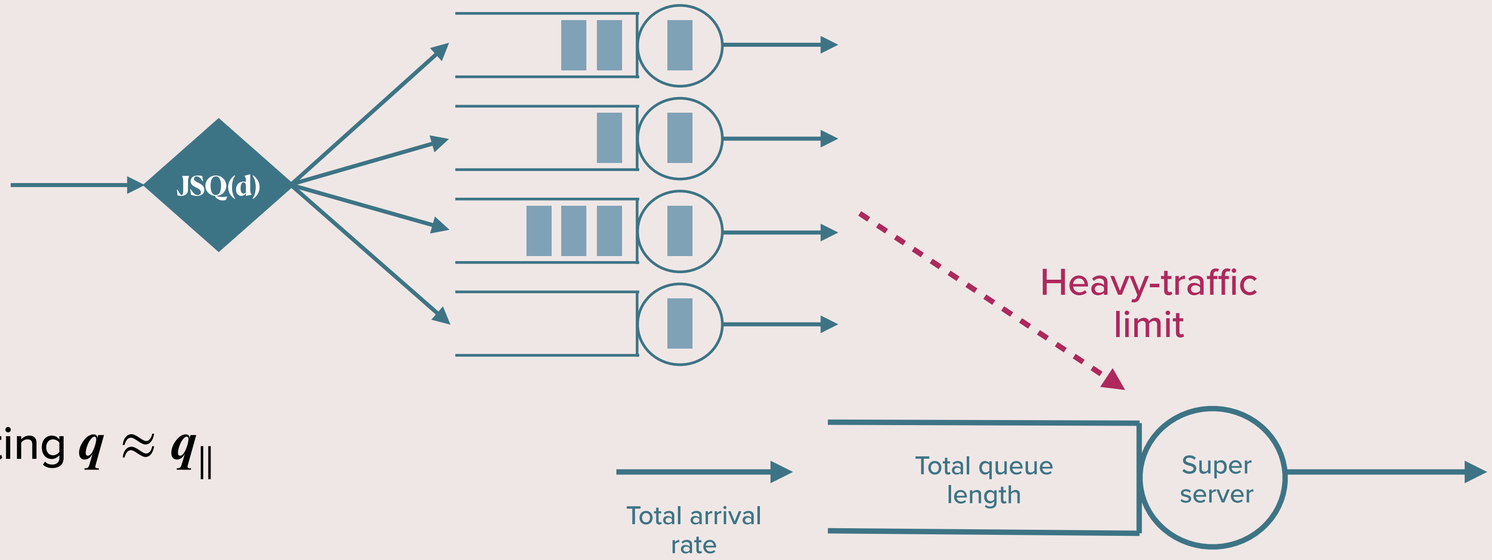


Formal Definition of CRP

Define:

$$\boldsymbol{q}_{\|} \triangleq \mathbf{1}\left(\frac{\sum q_i}{n}\right)$$

$\boldsymbol{q}_{\perp} \triangleq \boldsymbol{q} - \boldsymbol{q}_{\|}$ = Error of approximating $\boldsymbol{q} \approx \boldsymbol{q}_{\|}$



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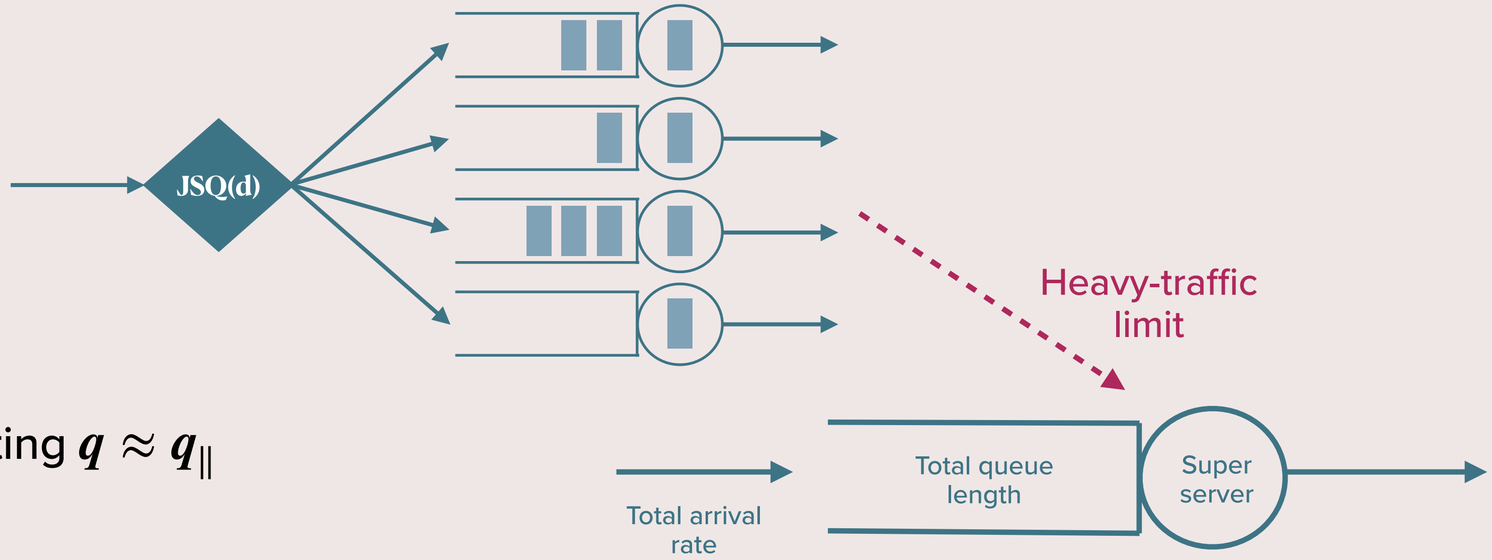

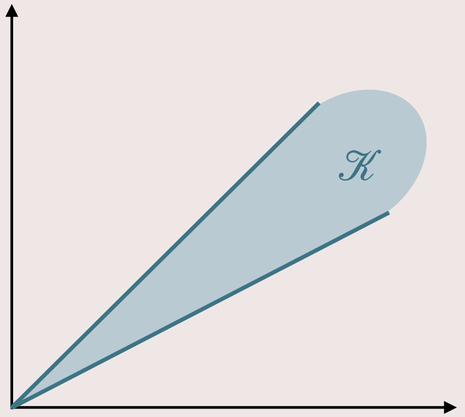

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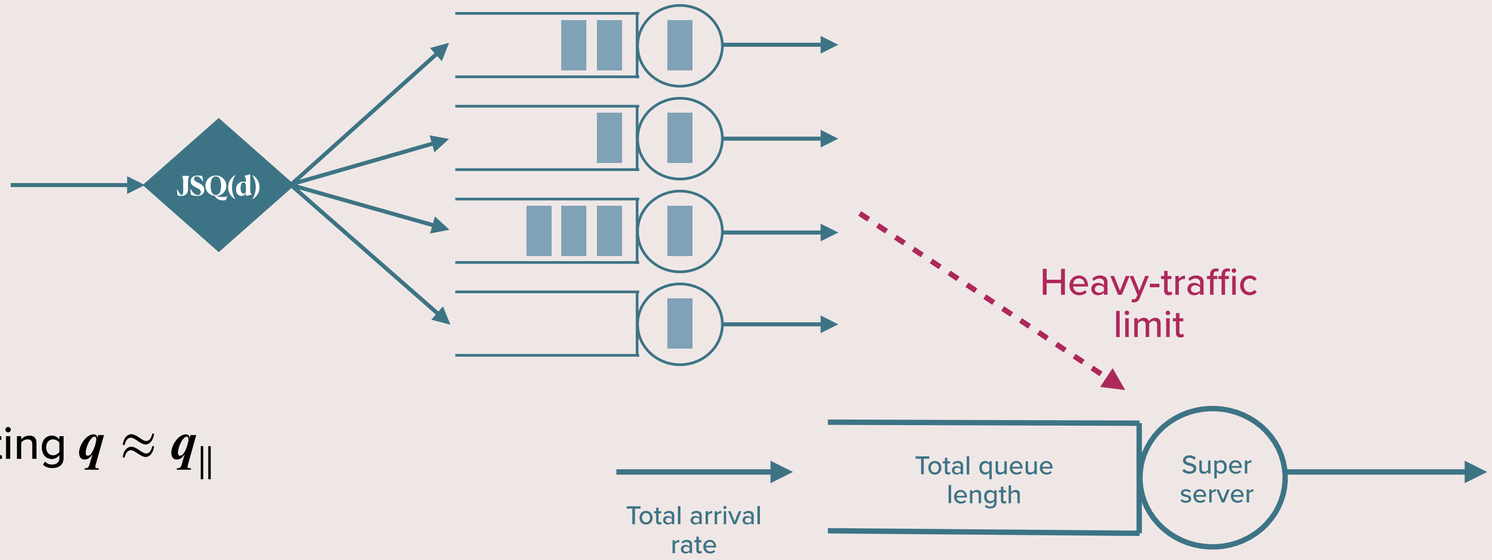

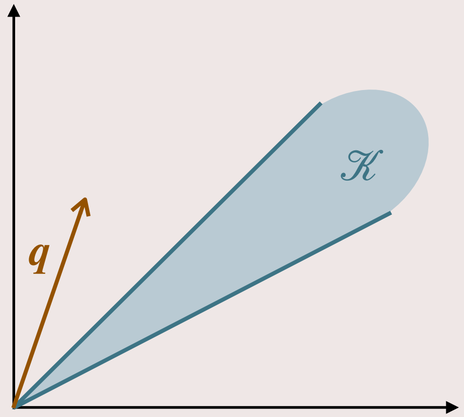

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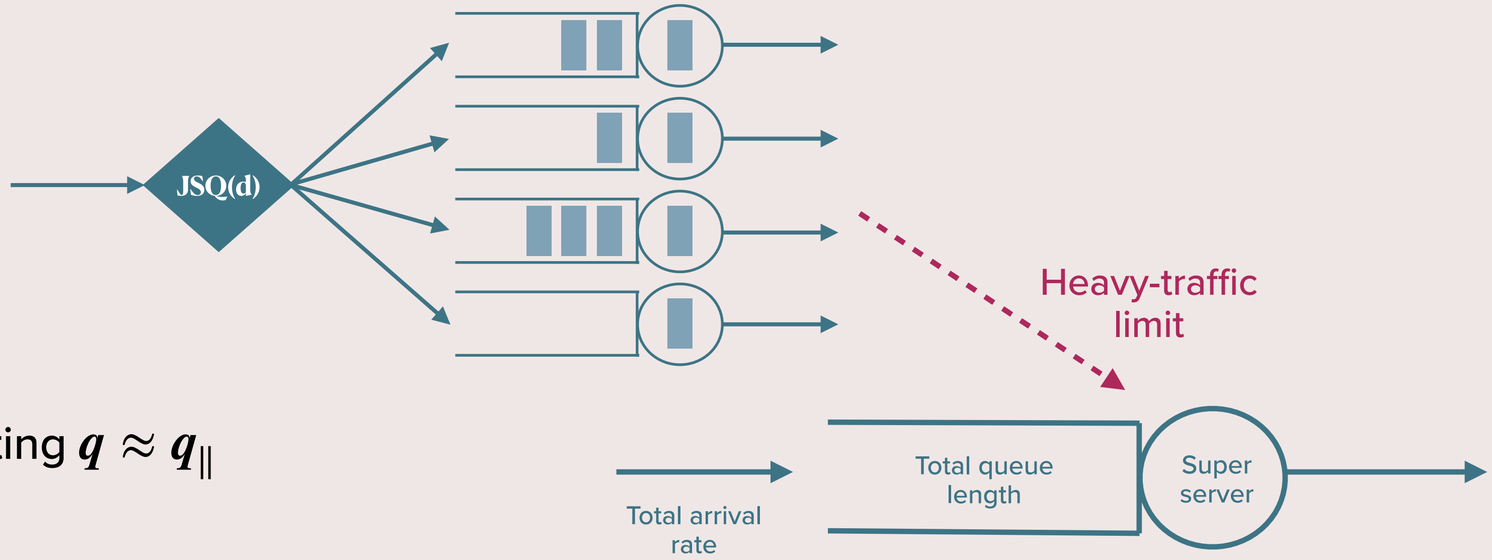

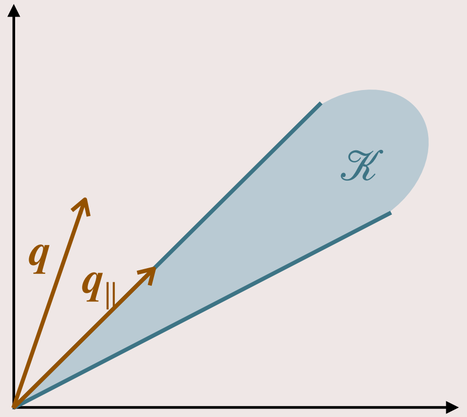

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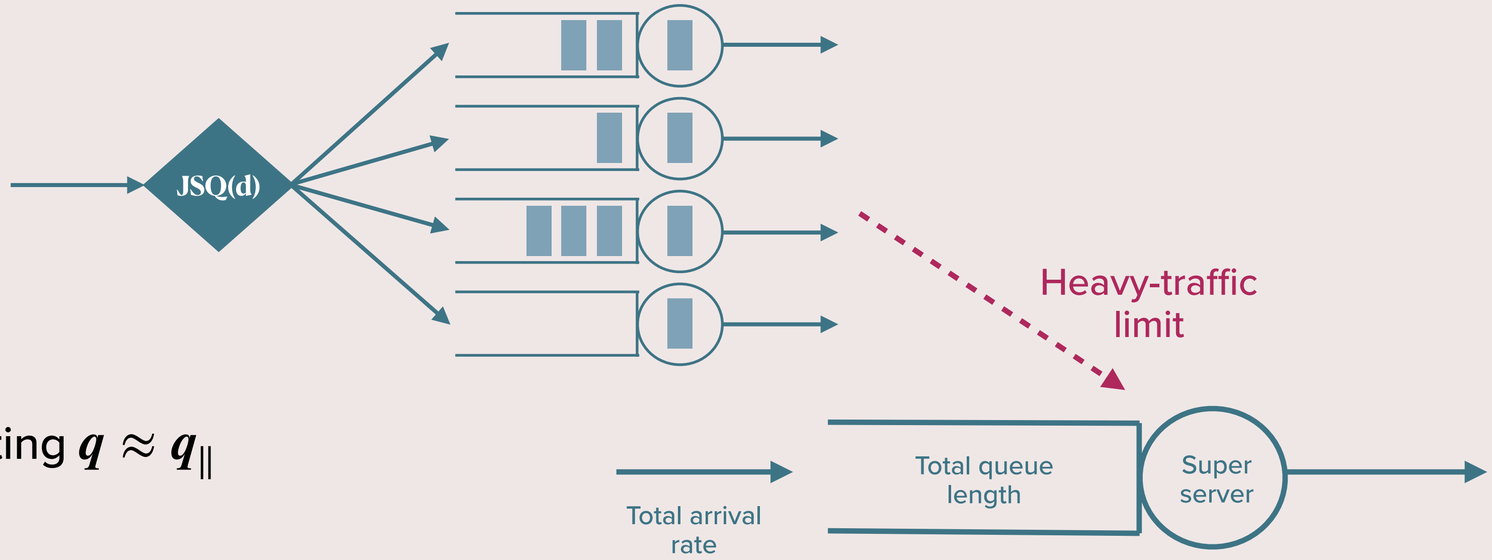

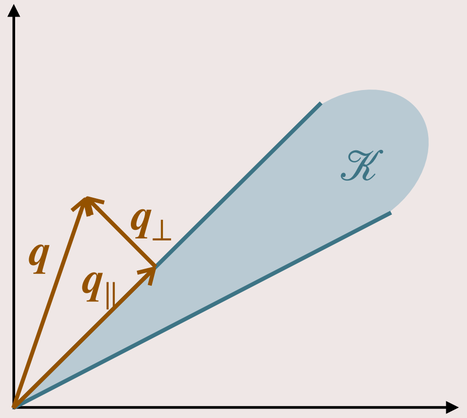

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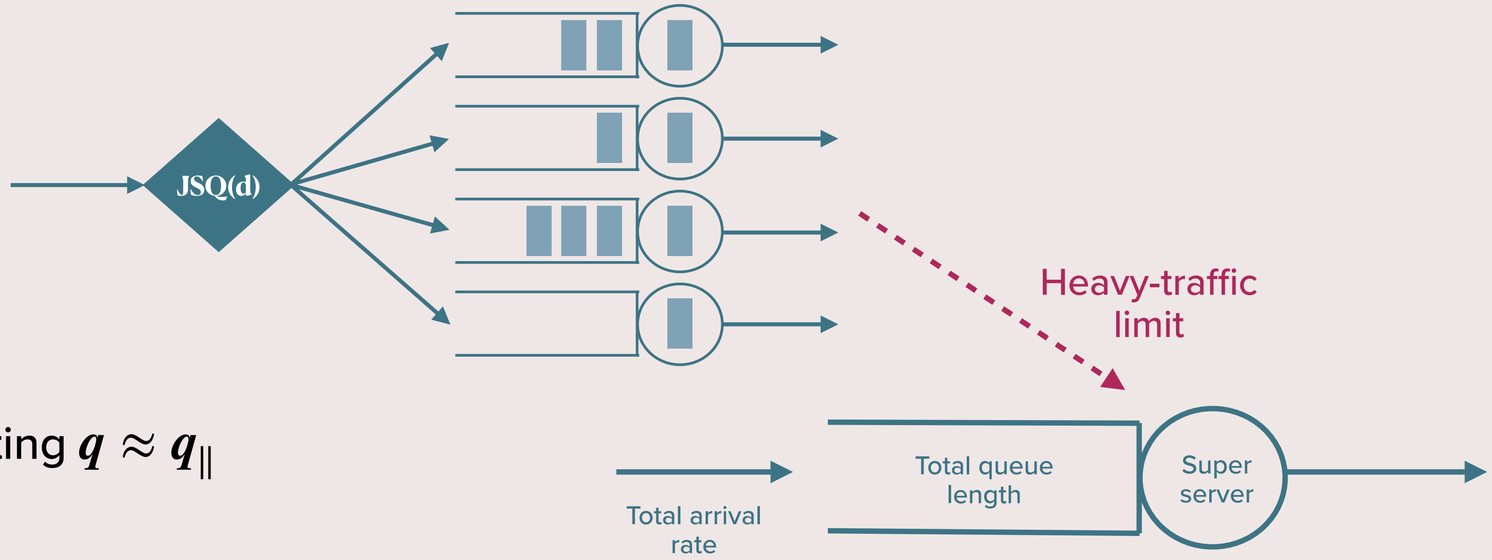

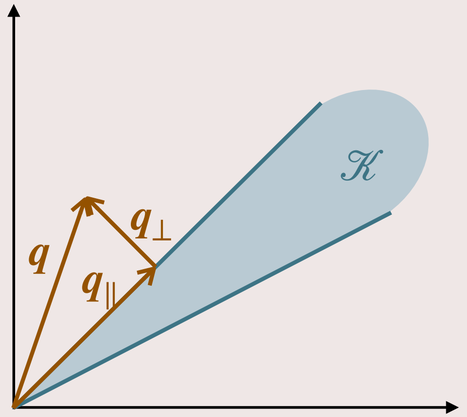

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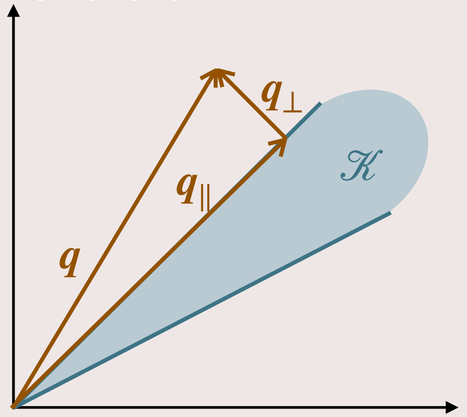

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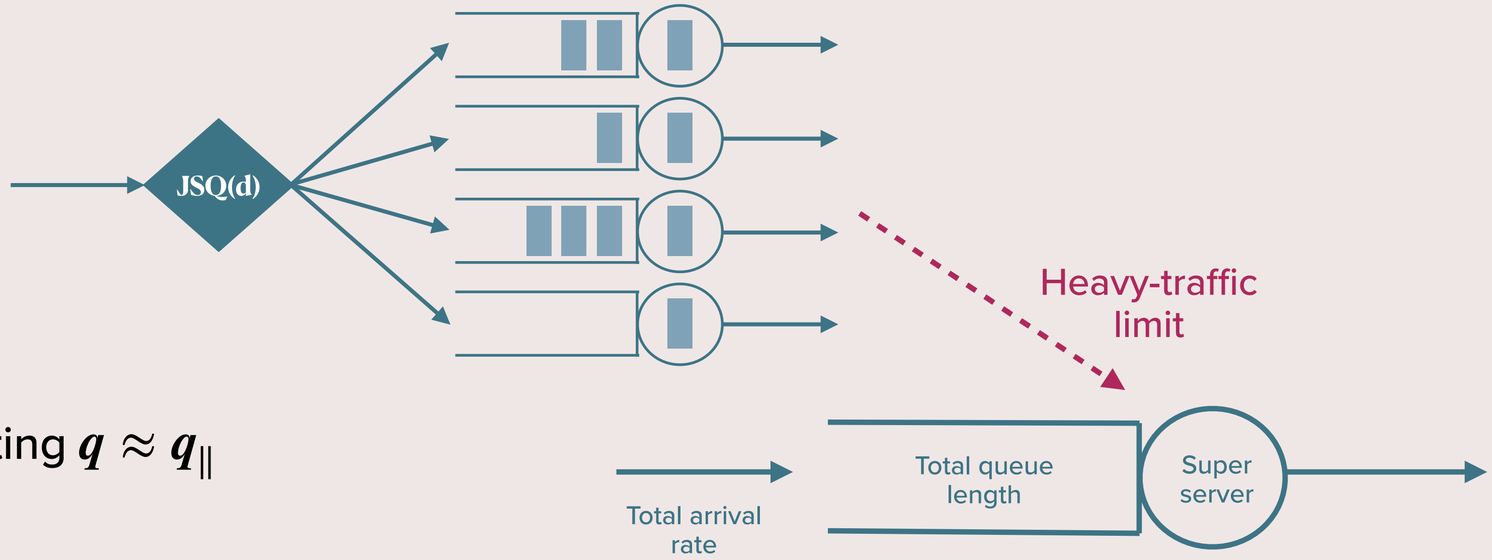

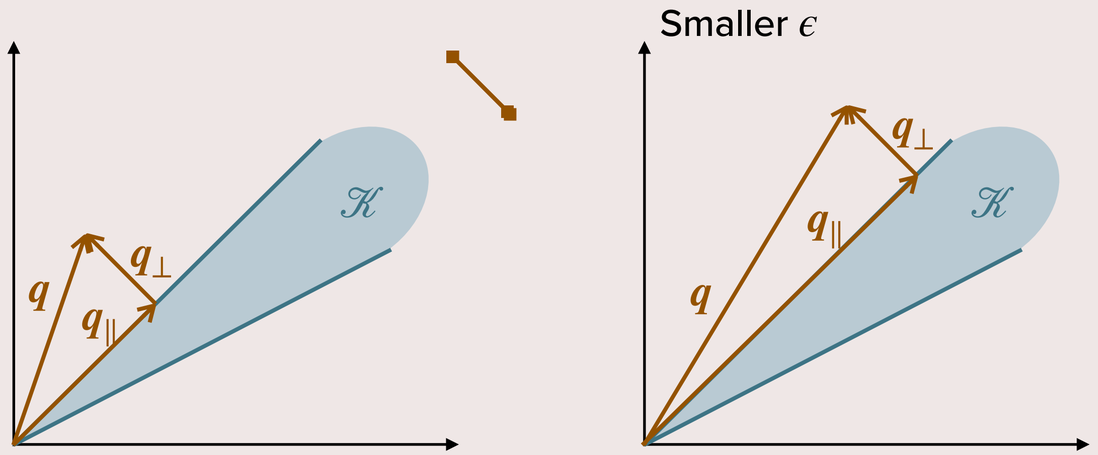

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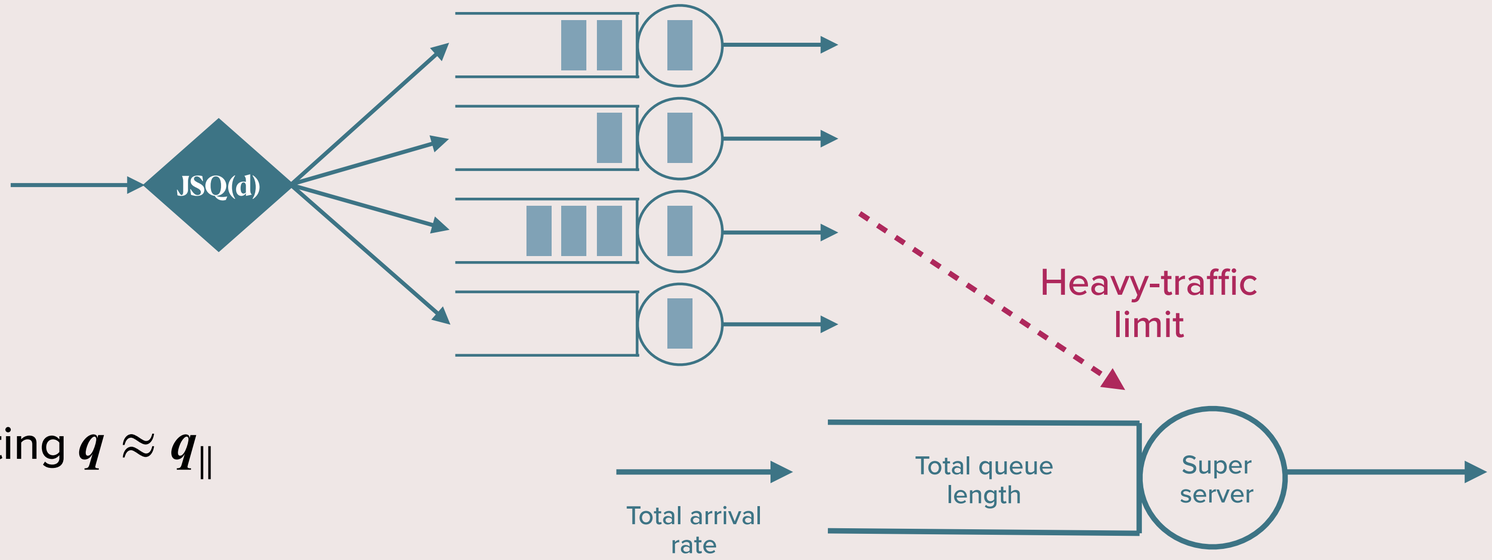

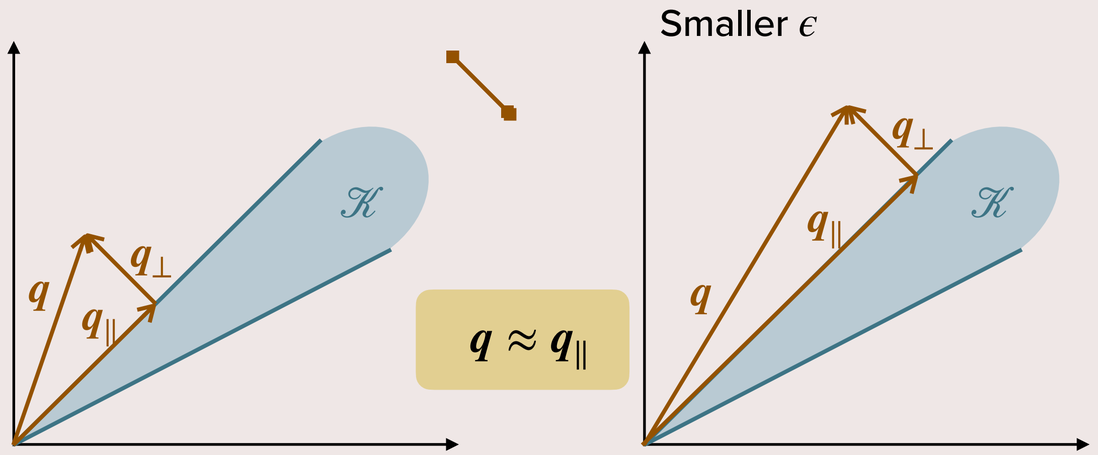

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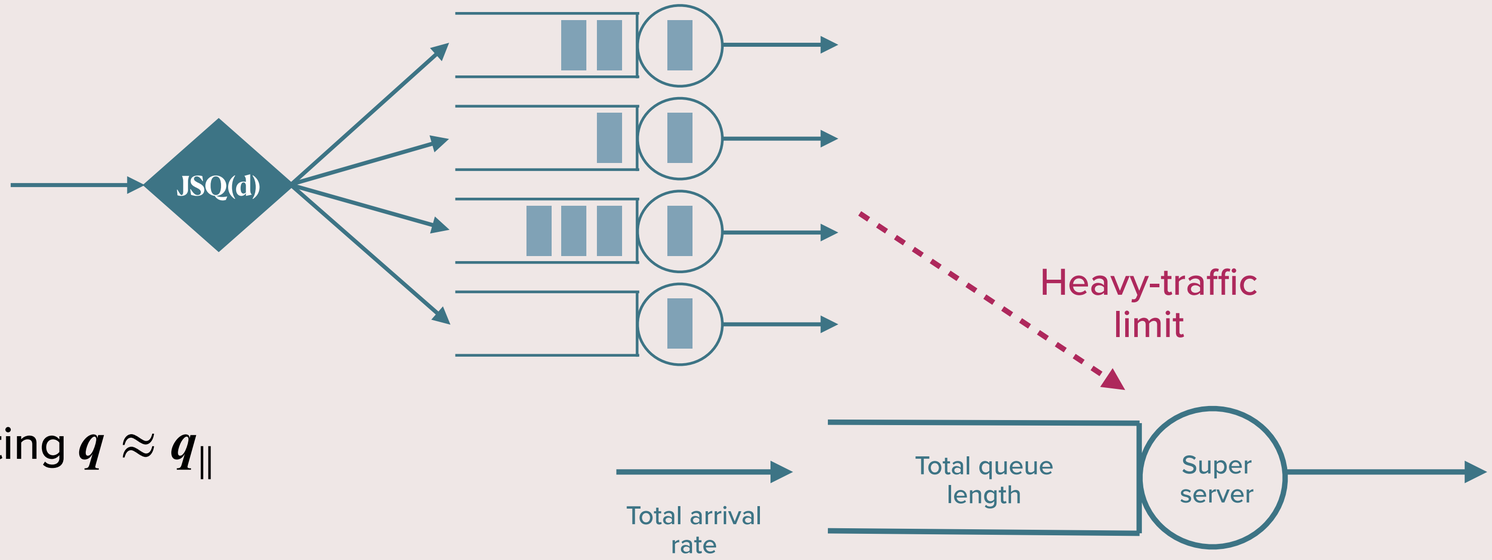
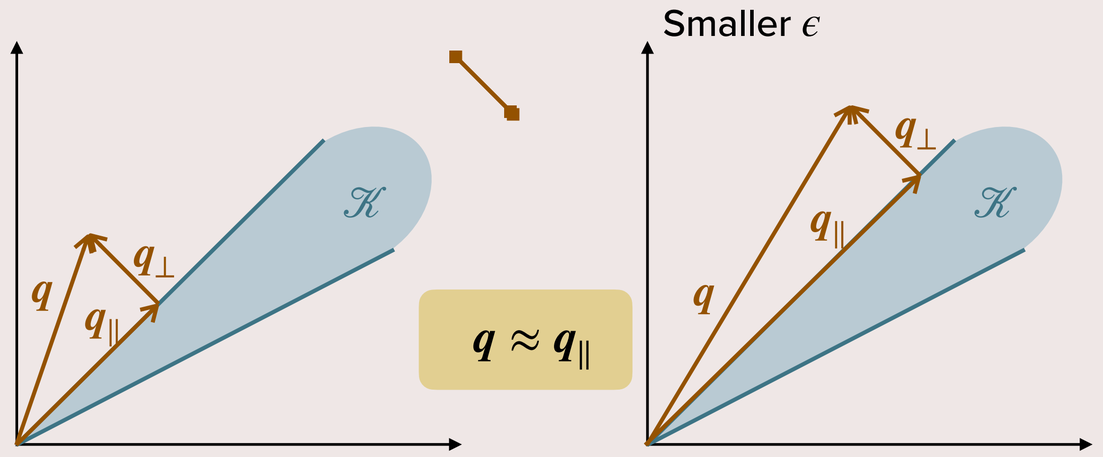


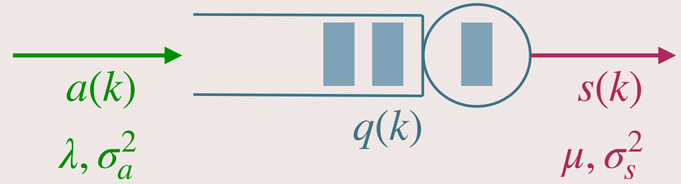
CRP:

$\boldsymbol{q}_{\perp}$ is negligible in heavy traffic

Formally, $\mathbb{E}\left[\|\boldsymbol{q}_{\perp}\|^m\right] \leq C_m$ for all $m = 1,2,3\ldots$



Single-Server Queue and Drift Method



Dynamics of the queues:

$$q(k+1) = q(k) + a(k) - s(k) + u(k)$$

$$\implies q(k+1)u(k) = 0$$

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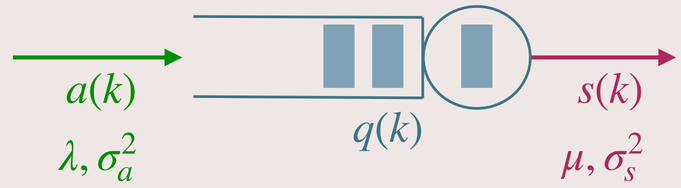
$$\mathbb{E}\left[q^2(k+1)\right] = \mathbb{E}\left[q^2(k)\right]$$



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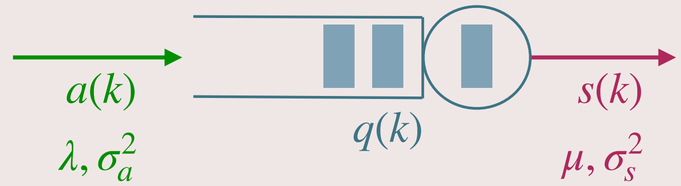

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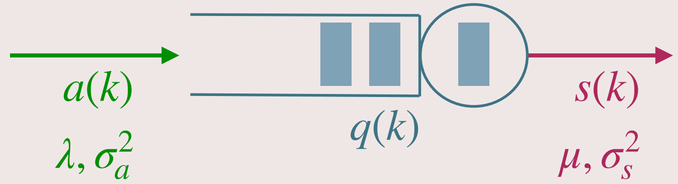

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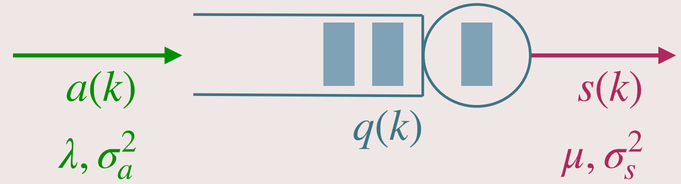

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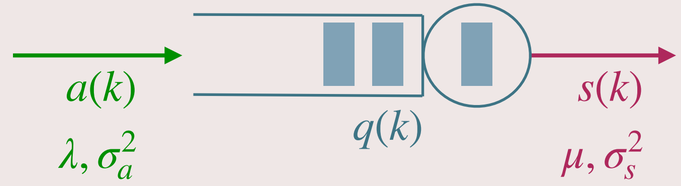

Dynamics of the queues:

$$q(k+1) = q(k) + a(k) - s(k) + u(k)$$

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Drift Method:

Test function:

Obtain:

Single-Server Queue and Drift Method

- In steady-state ($k \to \infty$):

$$\mathbb{E}\left[q^2(k+1)\right] = \mathbb{E}\left[q^2(k)\right]$$

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$$\mathbb{E}[q] = \frac{\mathbb{E}\left[(a-s)^2\right]}{2\epsilon} - \frac{\mathbb{E}\left[u^2\right]}{2\epsilon}$$

Small compared to the first term

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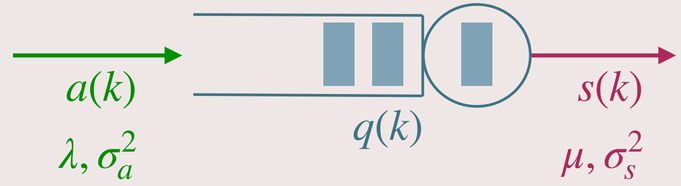

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Drift Method:

Test function: q^2

Obtain:

Single-Server Queue and Drift Method

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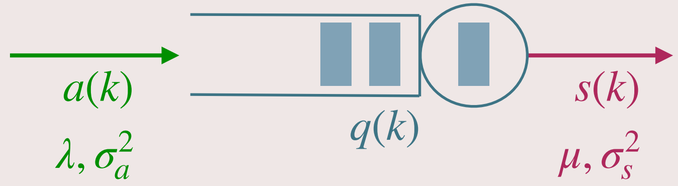

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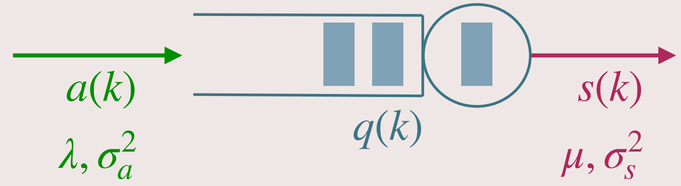

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Single-Server Queue and Drift Method

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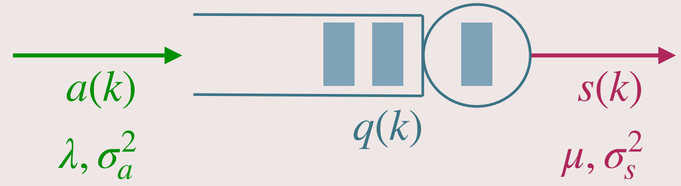

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Drift Method:

Test function: q^2 q^3

Obtain: $\mathbb{E}\left[\epsilon q\right]$

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Single-Server Queue and Drift Method

- In steady-state ($k \to \infty$):

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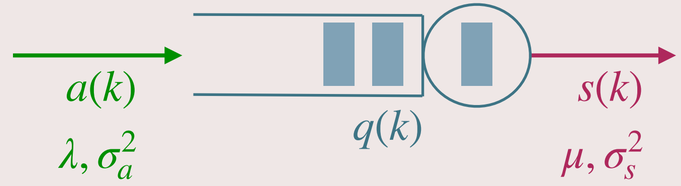

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Drift Method:

Test function: q^2, q^3

Obtain: $\mathbb{E}\left[\epsilon q\right]$, $\mathbb{E}\left[\epsilon^2 q^2\right]$

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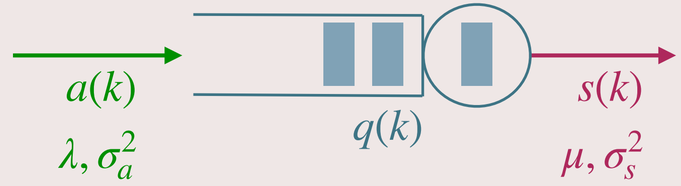

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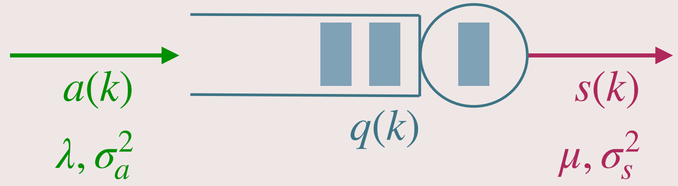

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Drift Method:

Test function: q^2, q^3, $\cdots$, q^m

Obtain:

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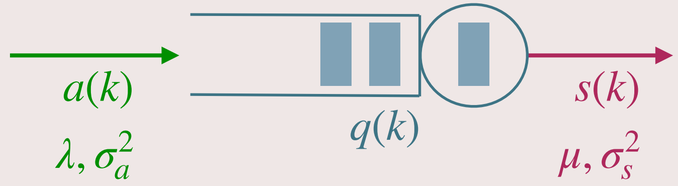

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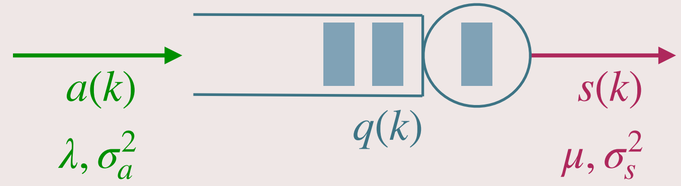

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$$\lim_{\epsilon \downarrow 0}(\quad) = \frac{\sigma_a^2 + \sigma_s^2}{2}, \quad 2\left(\frac{\sigma_a^2 + \sigma_s^2}{2}\right)^2, \quad \cdots, \quad (m-1)!\left(\frac{\sigma_a^2 + \sigma_s^2}{2}\right)^{m-1}$$

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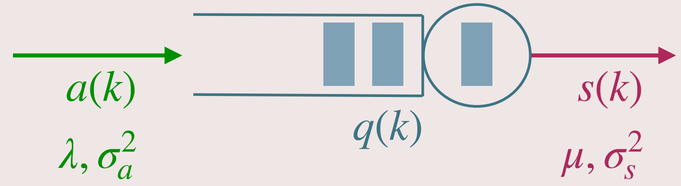

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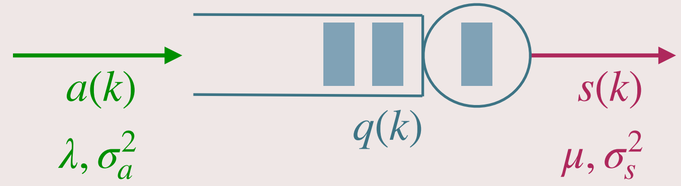

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Drift Method:

Test function: $1 + \epsilon q + \frac{1}{2}\epsilon^2 q^2 + \frac{1}{3!}\epsilon^3 q^3 + \cdots + \frac{1}{m!}\epsilon^m q^m + \cdots = e^{\epsilon q}$

Obtain: $1 + \mathbb{E}\left[\epsilon q\right] + \frac{1}{2}\mathbb{E}\left[\epsilon^2 q^2\right] + \cdots + \frac{1}{(m-1)!}\mathbb{E}\left[\epsilon^{m-1} q^{m-1}\right] + \cdots = \mathbb{E}\left[e^{\epsilon q}\right]$

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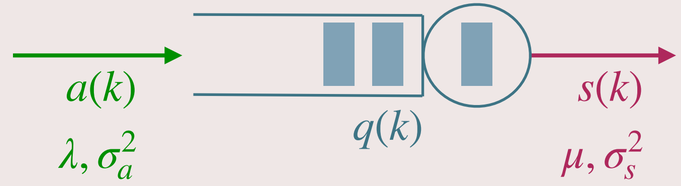

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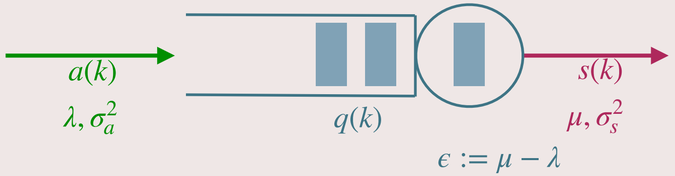
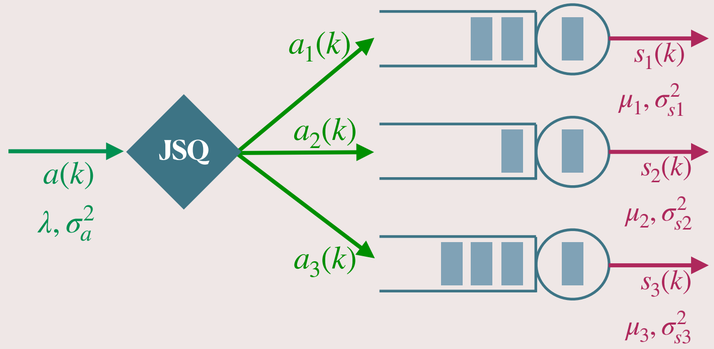
$$\epsilon q \Rightarrow Expo\left(\frac{\sigma_a^2 + \sigma_s^2}{2}\right)$$

Use $e^{\theta \epsilon q}$ as test function

The MGF Method [HL, Maguluri '20]





$$\epsilon := \sum_i \mu_i - \lambda$$

$$\boldsymbol{q}_{\|} := \mathbf{1}\left(\frac{\sum_j q_j}{N}\right)$$

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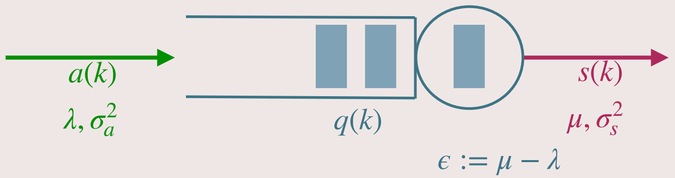

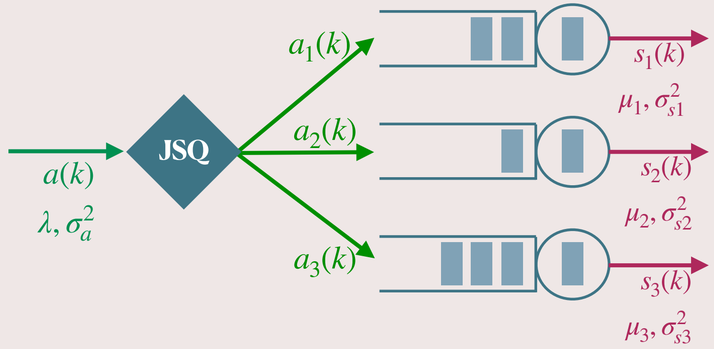

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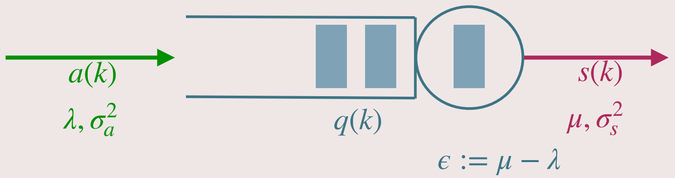

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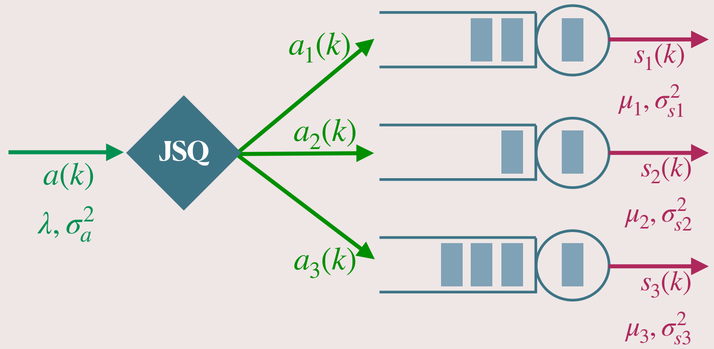

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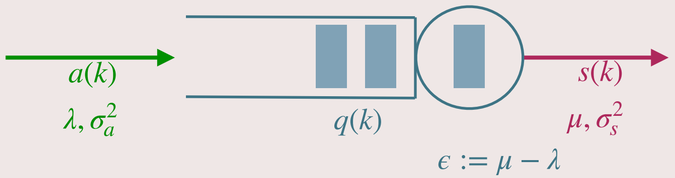

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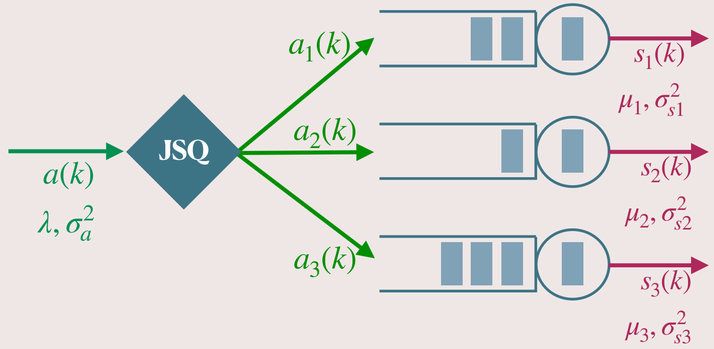

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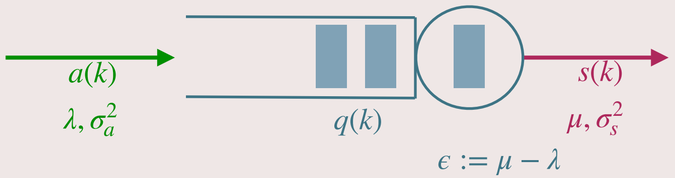

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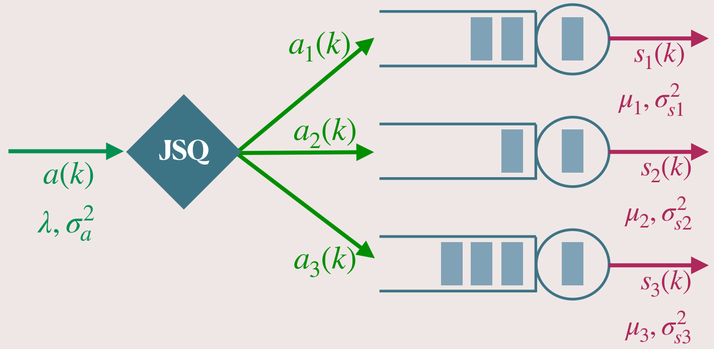

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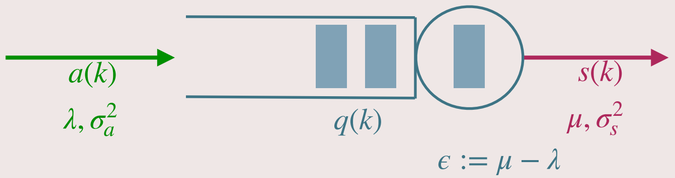

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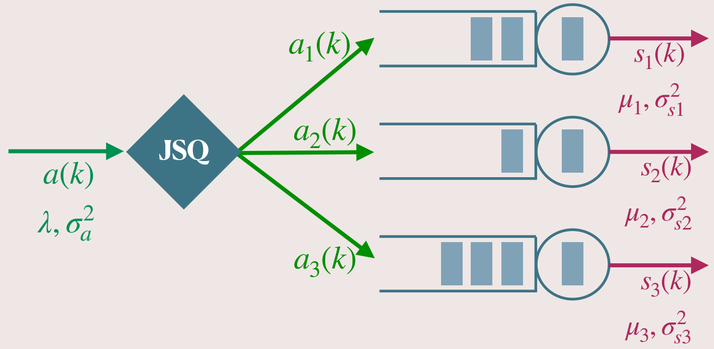

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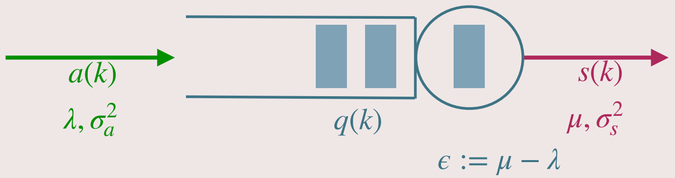

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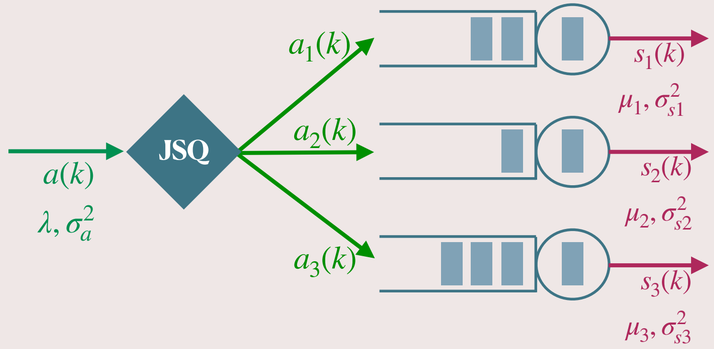

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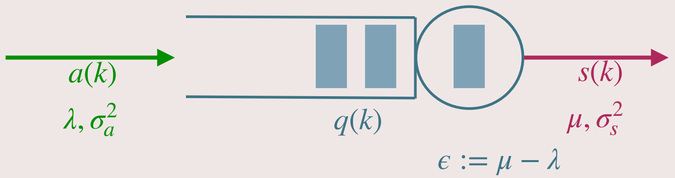

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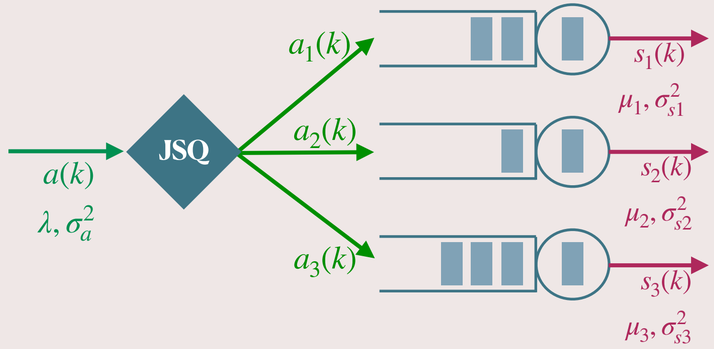

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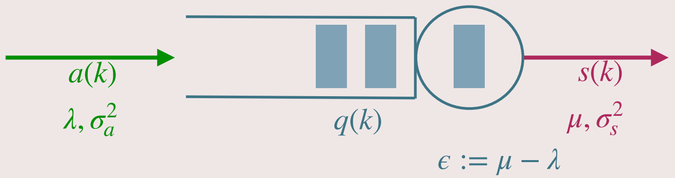

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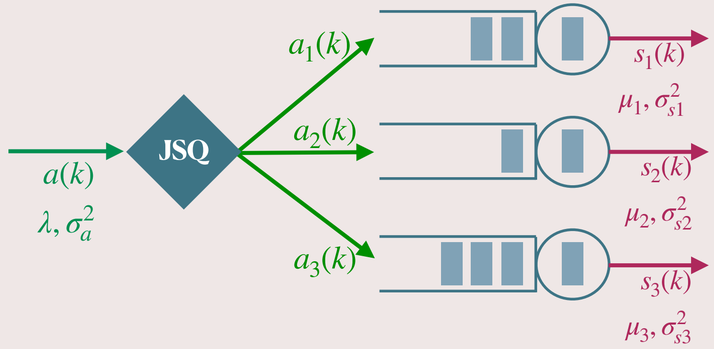

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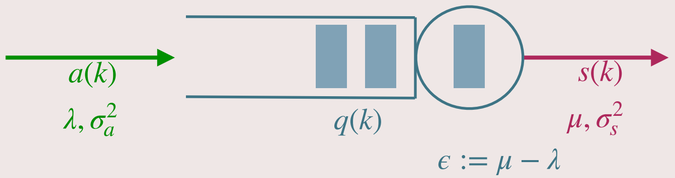

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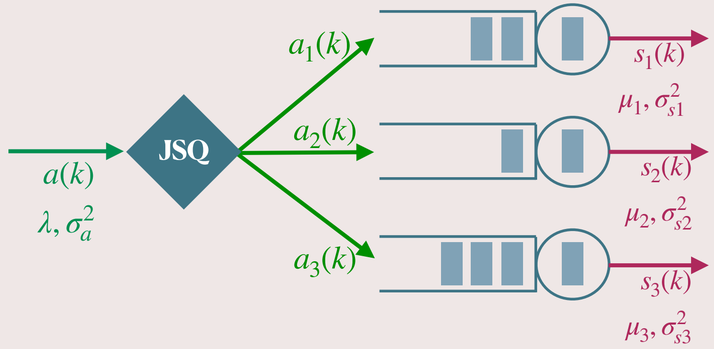

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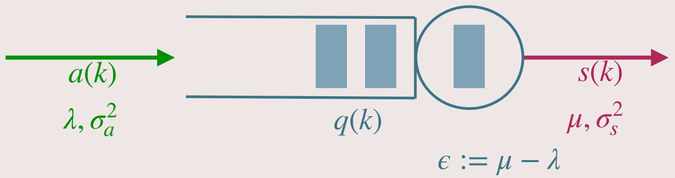

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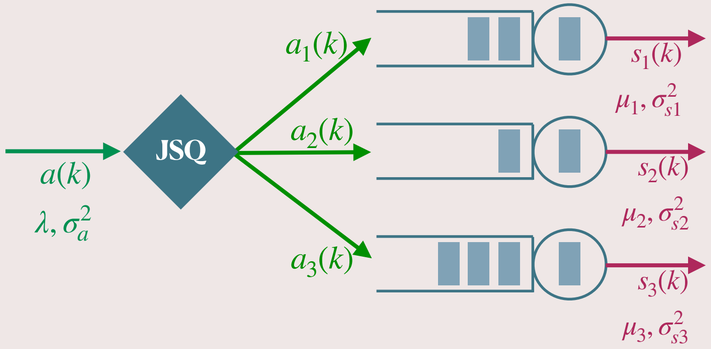

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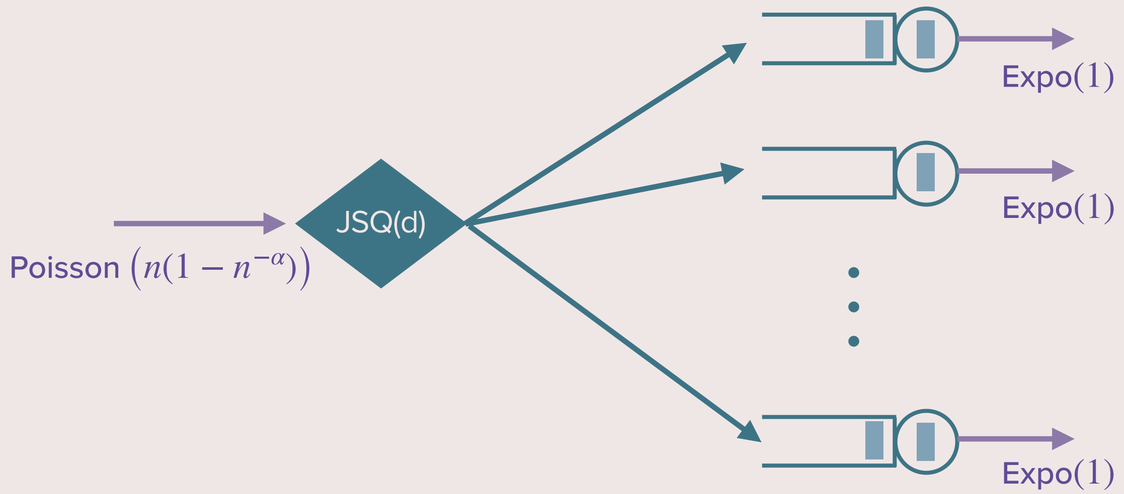
Proof Sketch

Step 1: Multiplicative State Space Collapse



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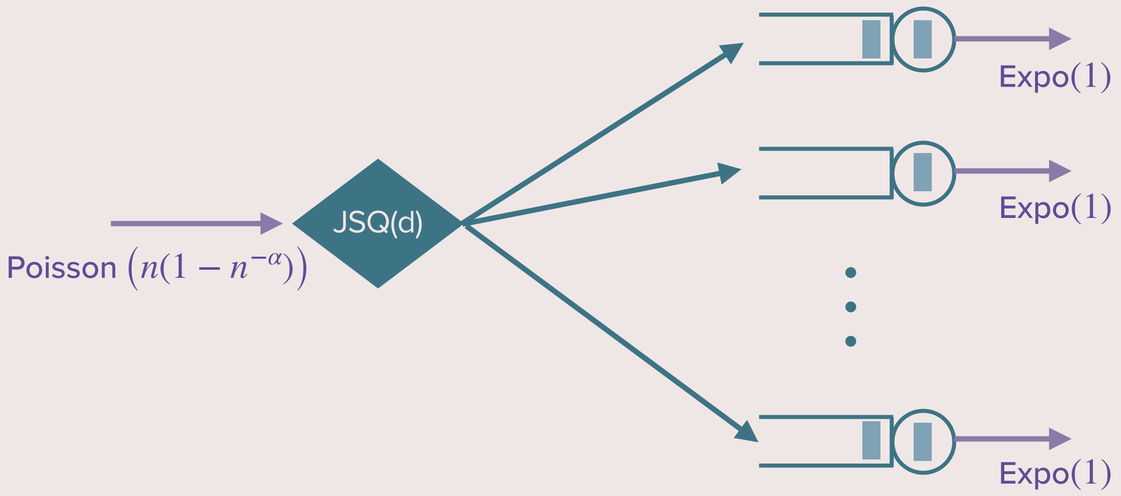
Proposition (SSC): [HL, Maguluri '21]

$\boldsymbol{q}_{\|}(k) = \mathbf{1}\left(\frac{\sum_i q_i}{n}\right)$ = Projection of $\boldsymbol{q}$ on $\mathbf{1}$

$\boldsymbol{q}_{\perp}(k) = \boldsymbol{q}(k) - \boldsymbol{q}_{\|}(k)$ error of approximating $\boldsymbol{q} \approx \boldsymbol{q}_{\|}$

Then, for all $j = 1,2,3\ldots$ $\mathbb{E}\left[\|\boldsymbol{q}_{\perp}\|^j\right]^{\frac{1}{j}} \leq Cj\left(\frac{n^2}{d-1}\right)$



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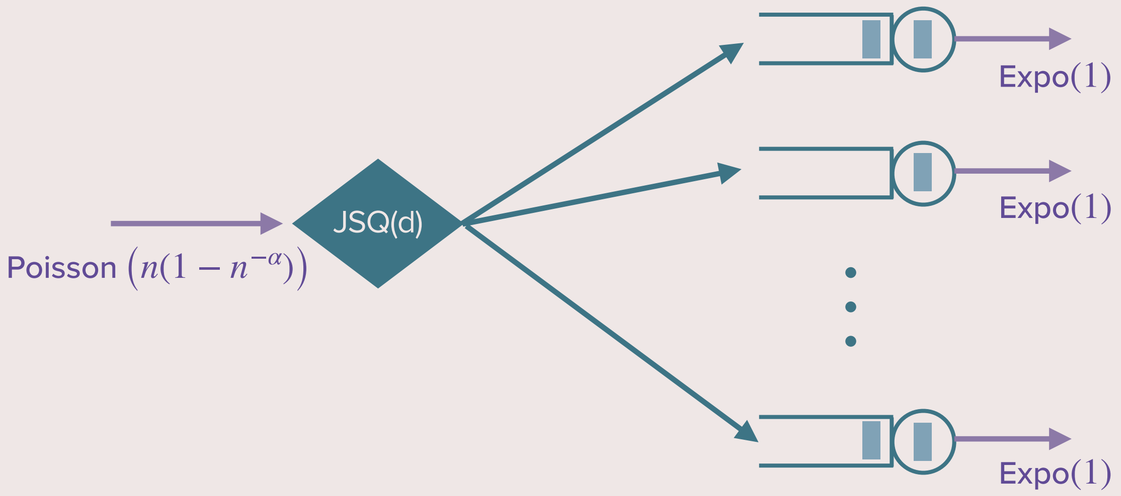

Step 2: Convergence in distribution

Proof Sketch

Step 1: Multiplicative State Space Collapse

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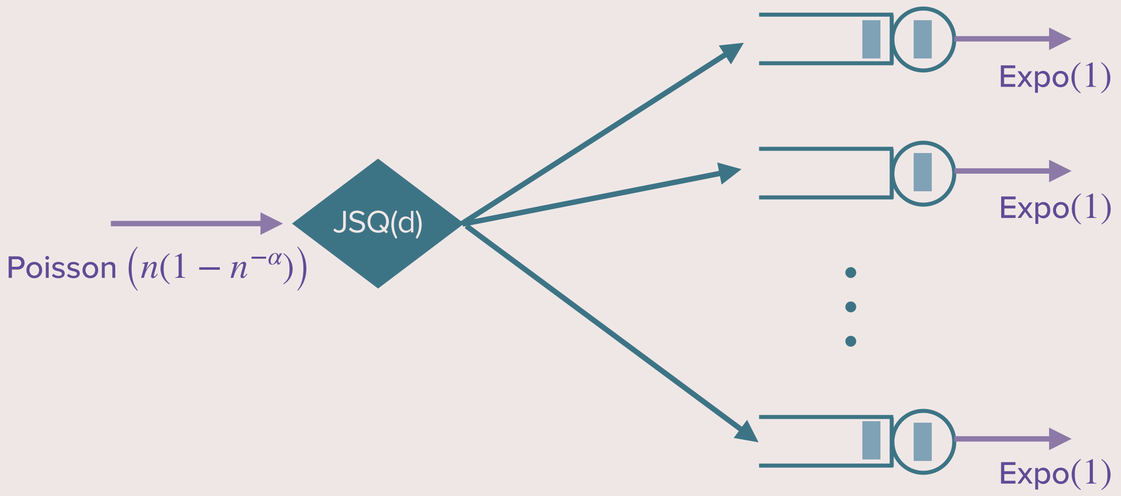

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Proof #1: Transform method in continuous time

- Extend method to continuous time
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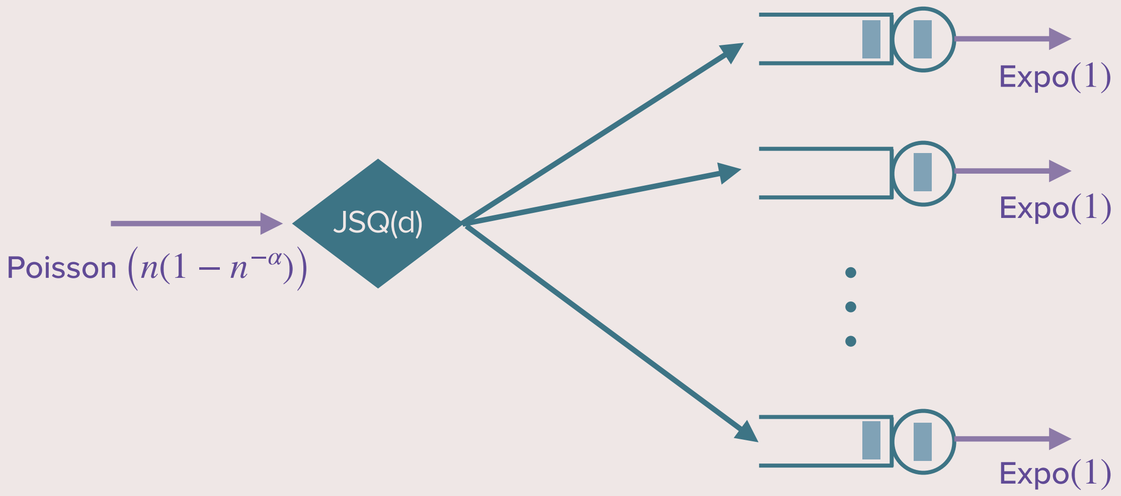

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Proof #2: Stein's method

- Bound Wasserstein's distance
- Obtain rate of convergence