Transform Method for Markov-Modulated Queues

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Northwestern University

YEQT 2025 — November 3-5th, 2025

Waiting in Line

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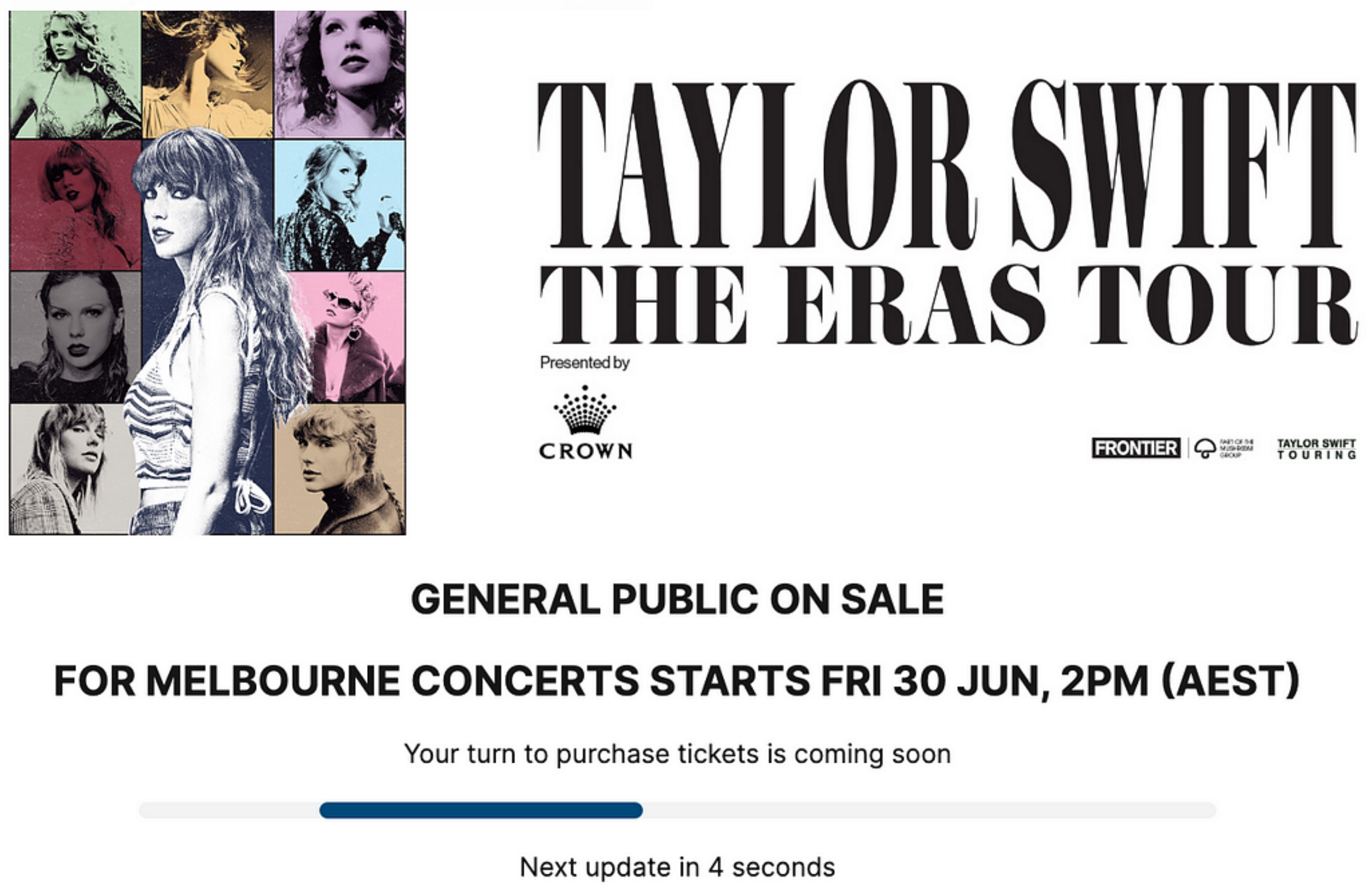

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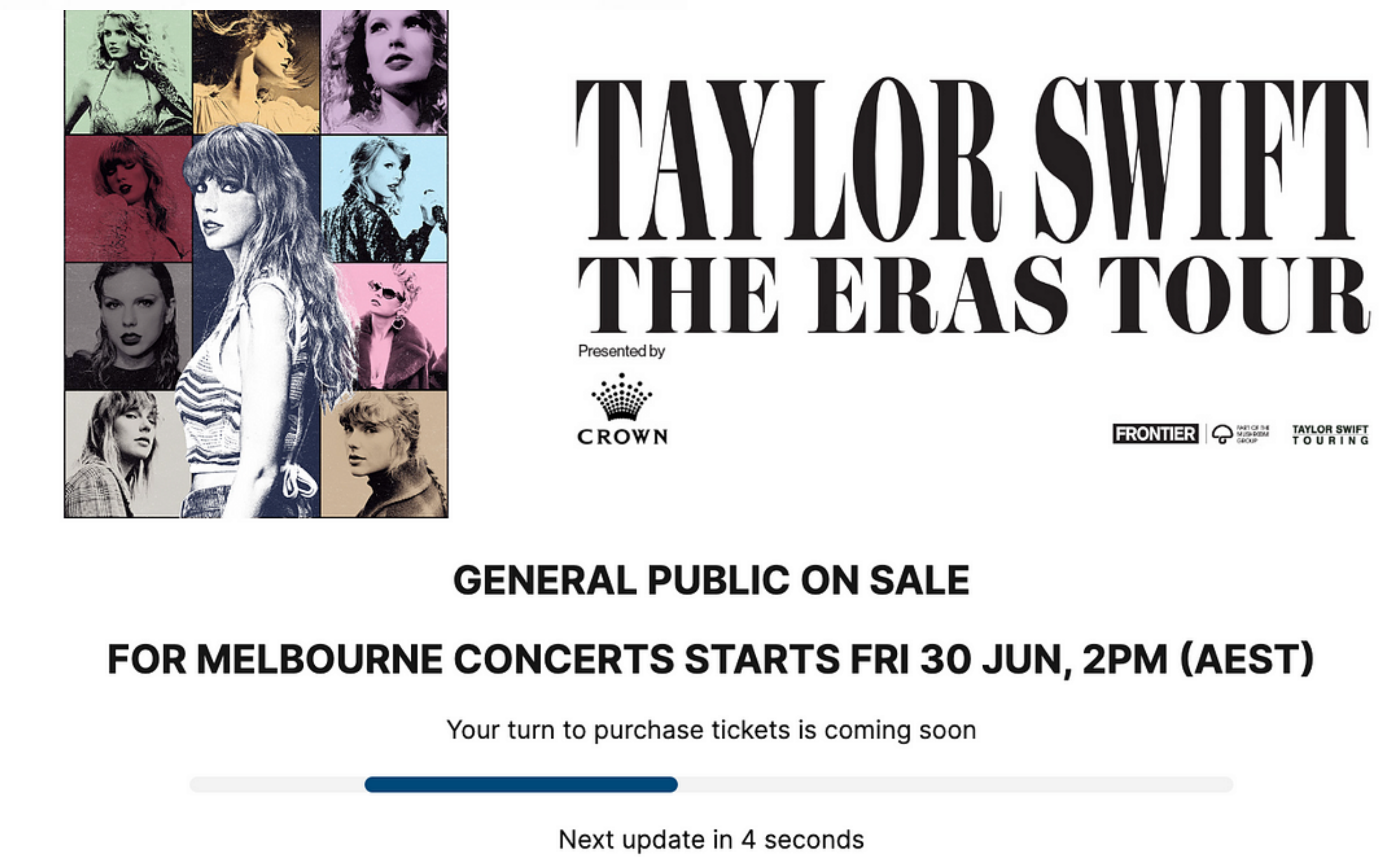

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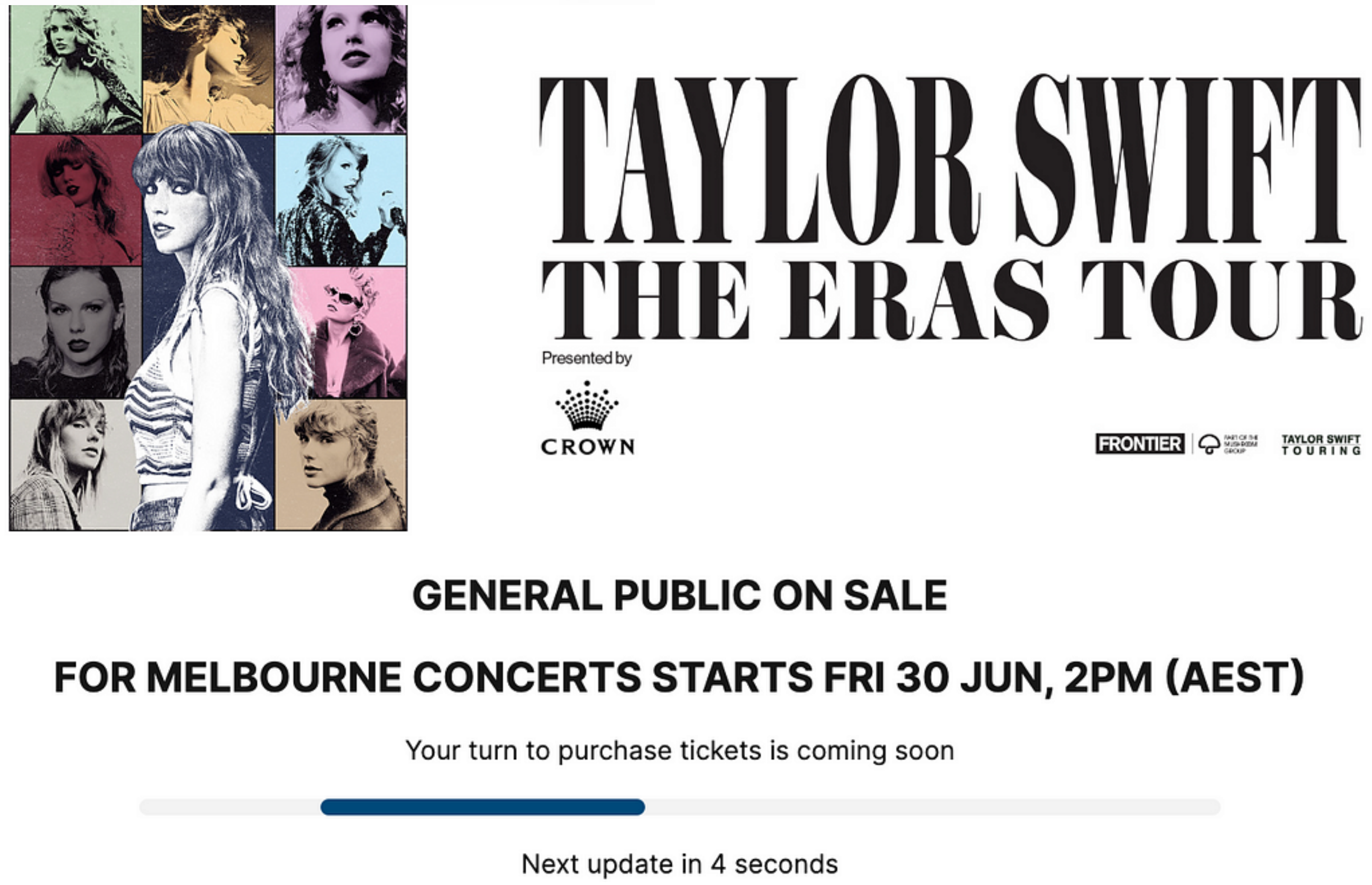

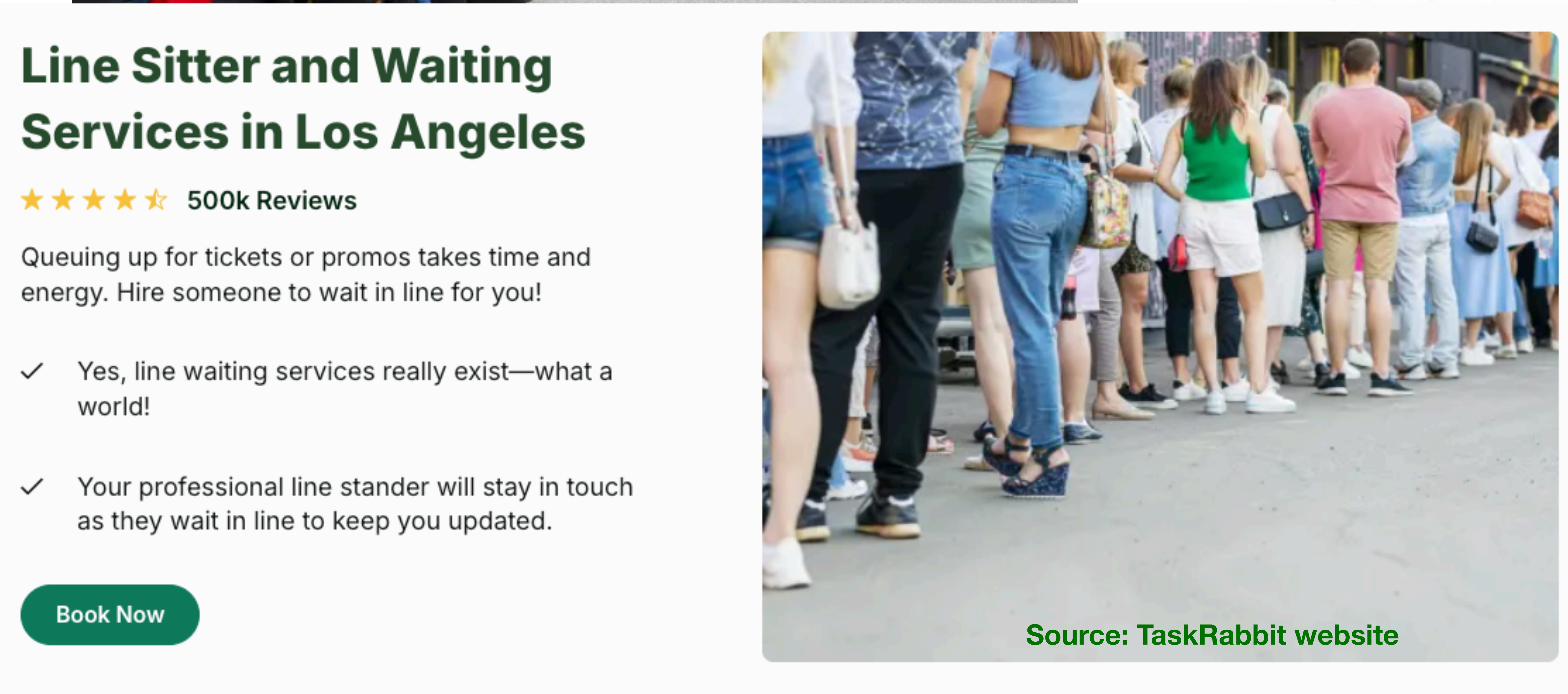

Source: TaskRabbit website

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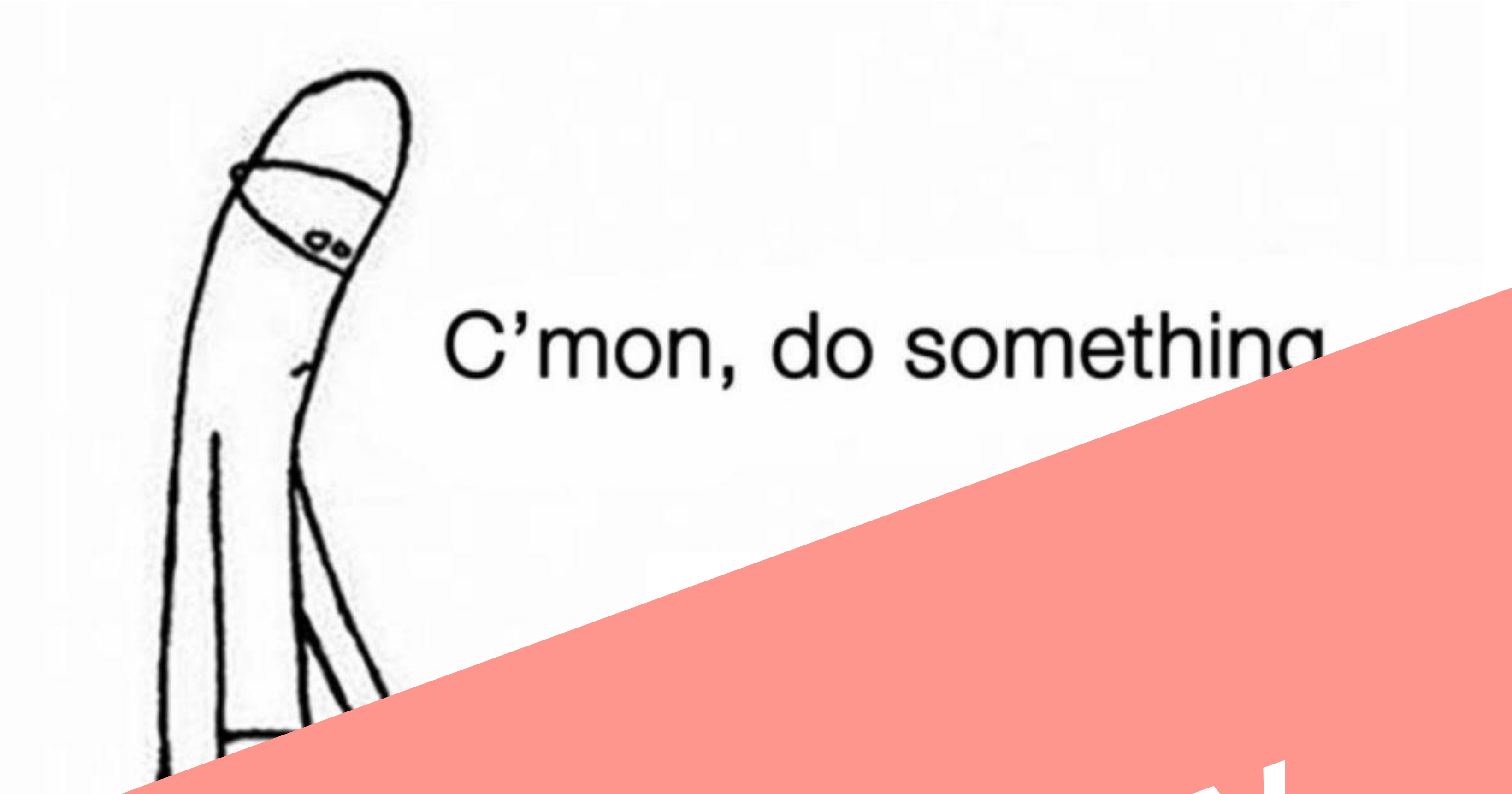

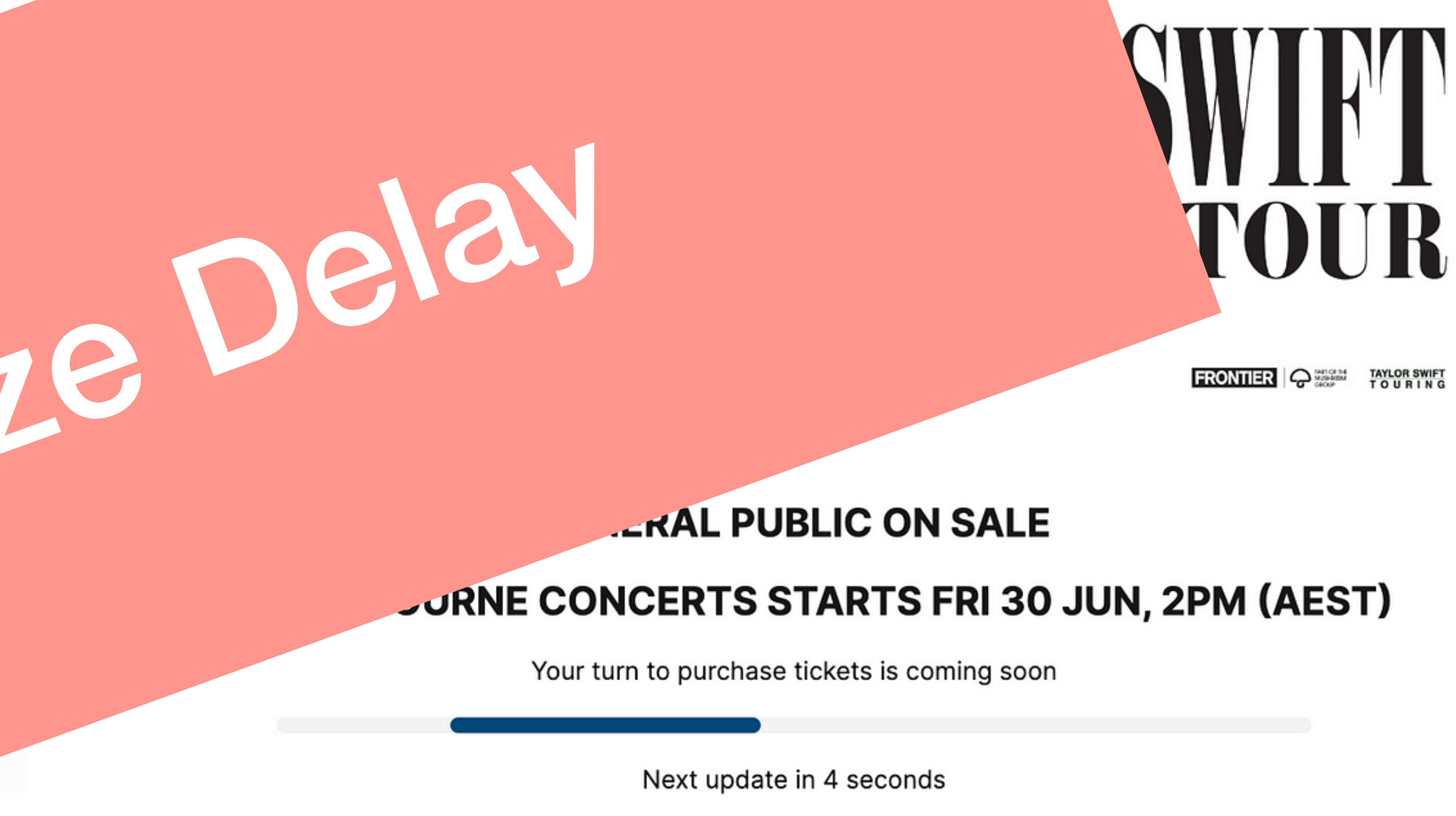

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Ultimate goal: Minimize Delay

Understanding Delay

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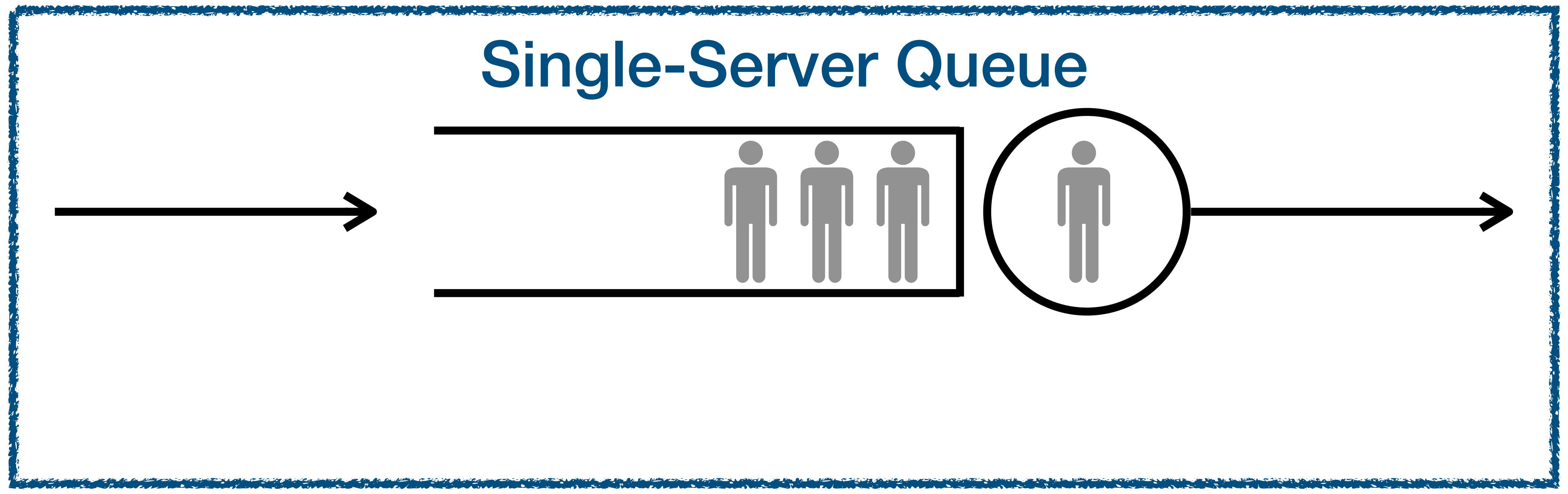

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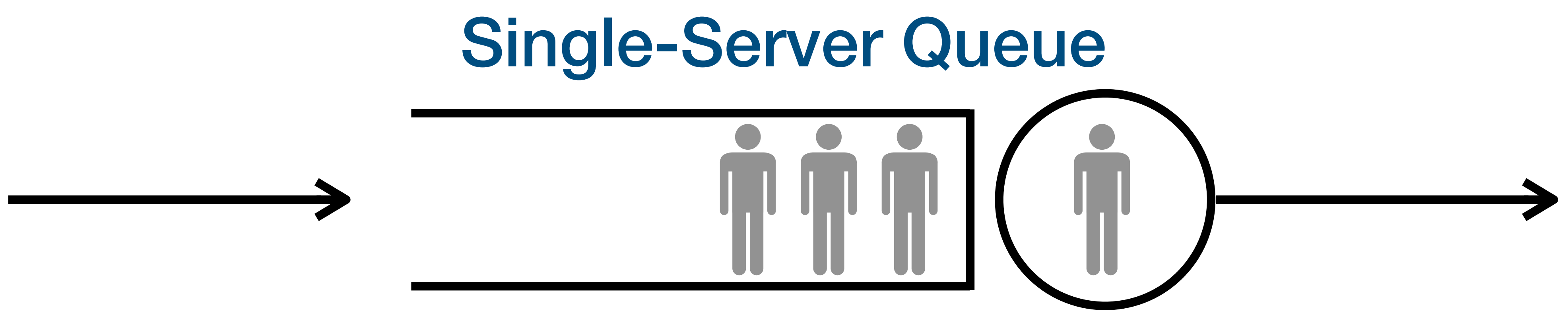
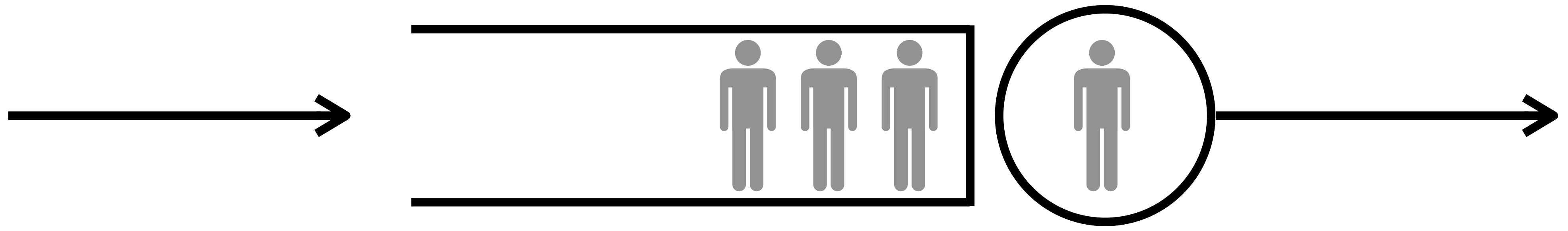


✓ Well understood
Distribution of q is known

Understanding Delay

Single-Server Queue



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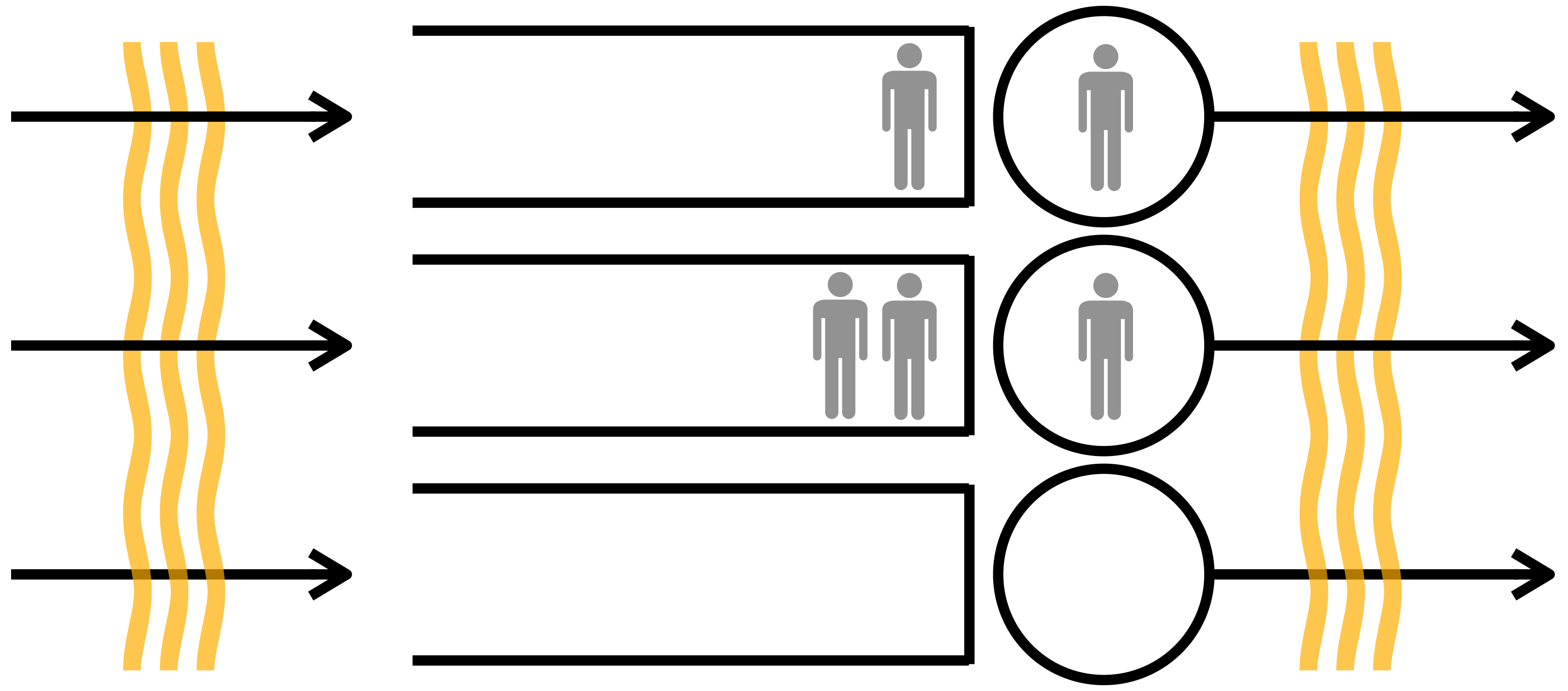
Multiple queues with correlated arrivals and services



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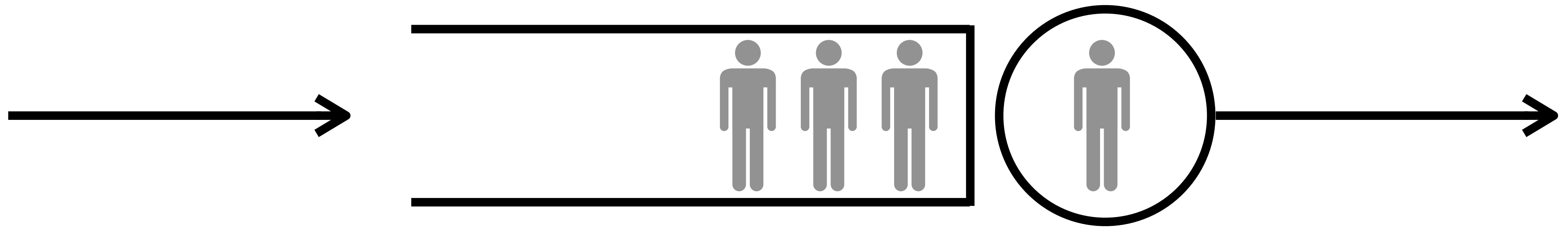

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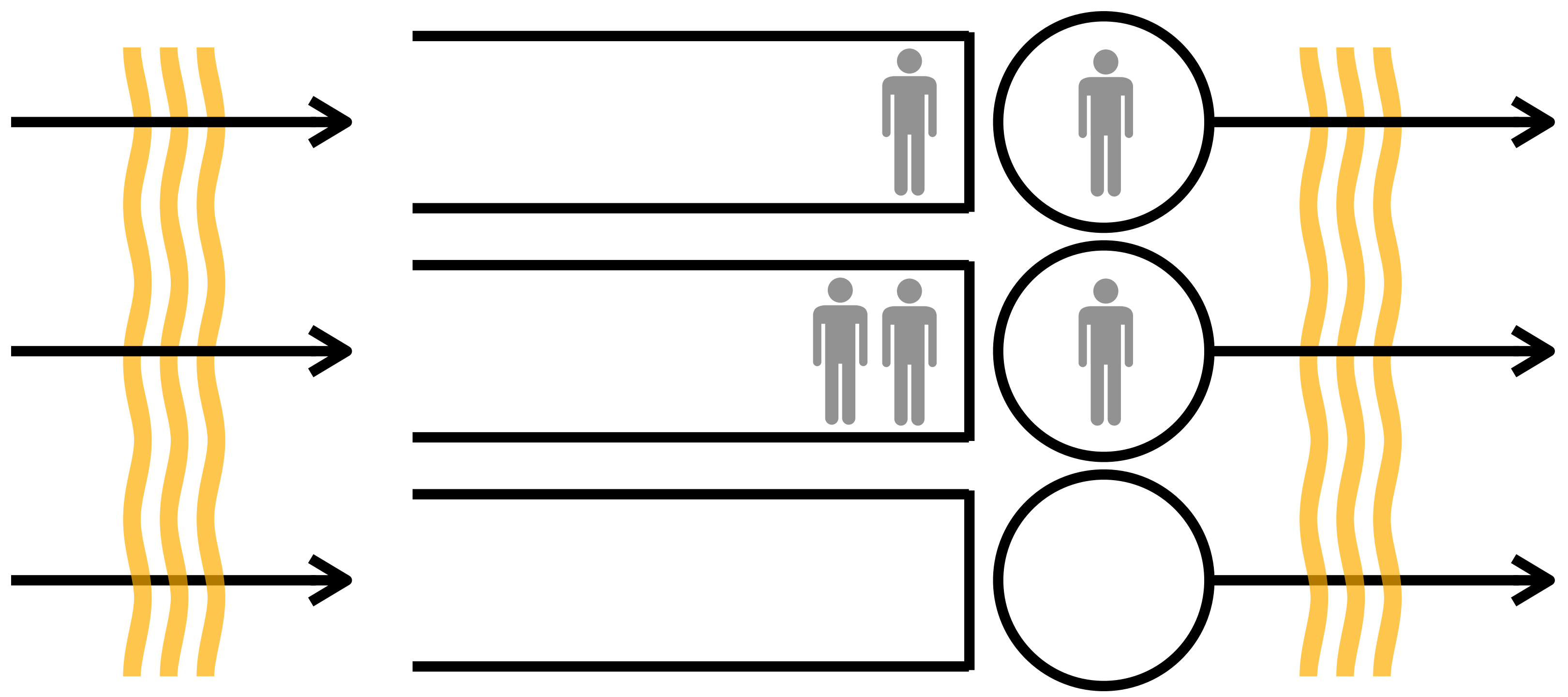
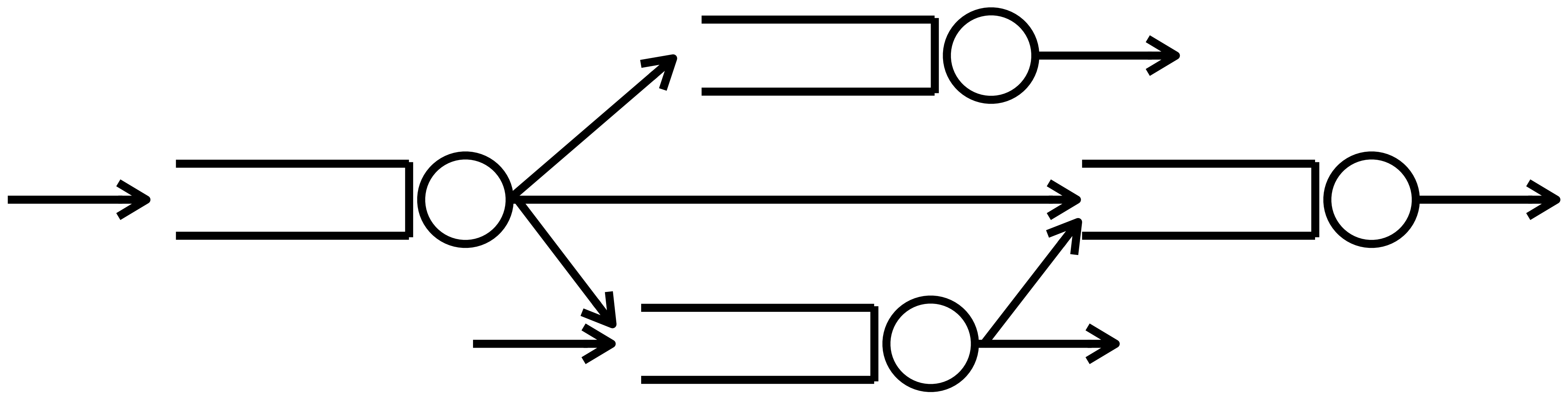


Networks of queues



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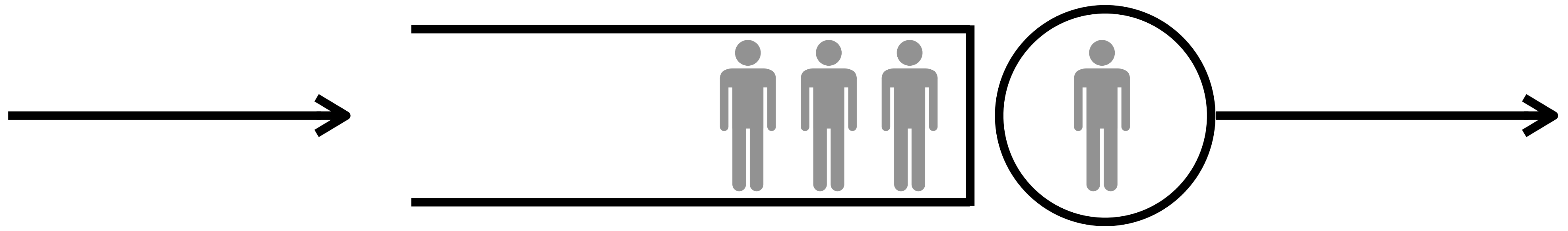

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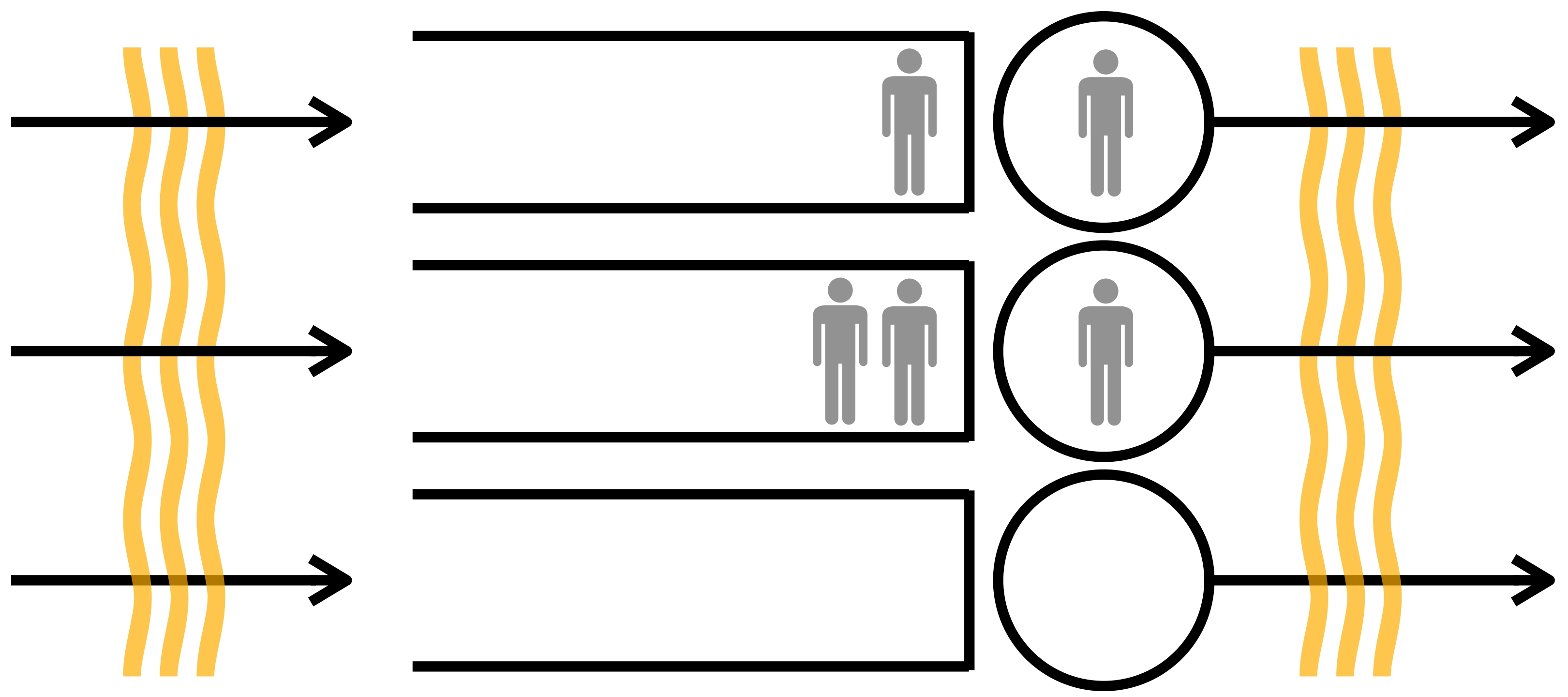

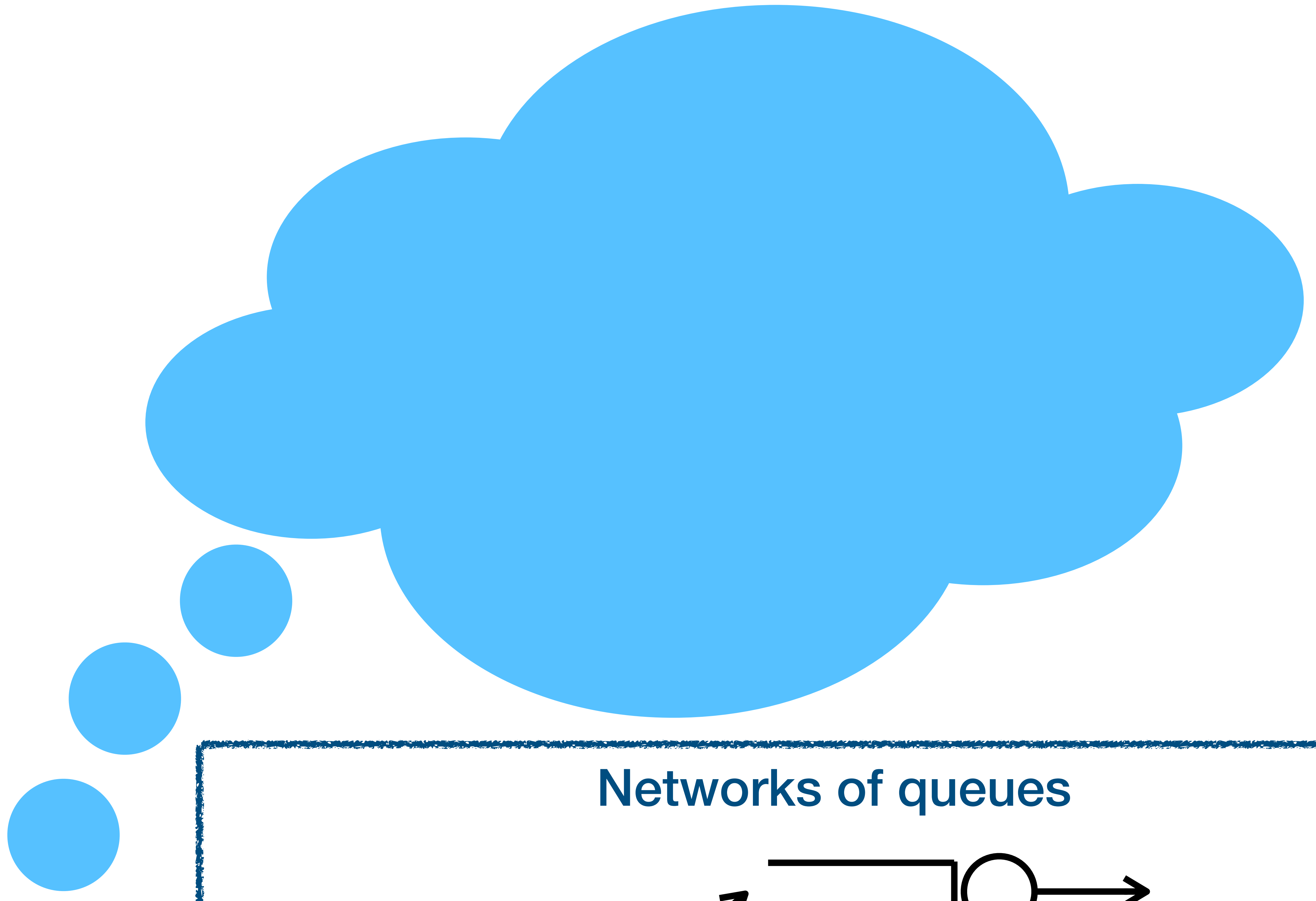

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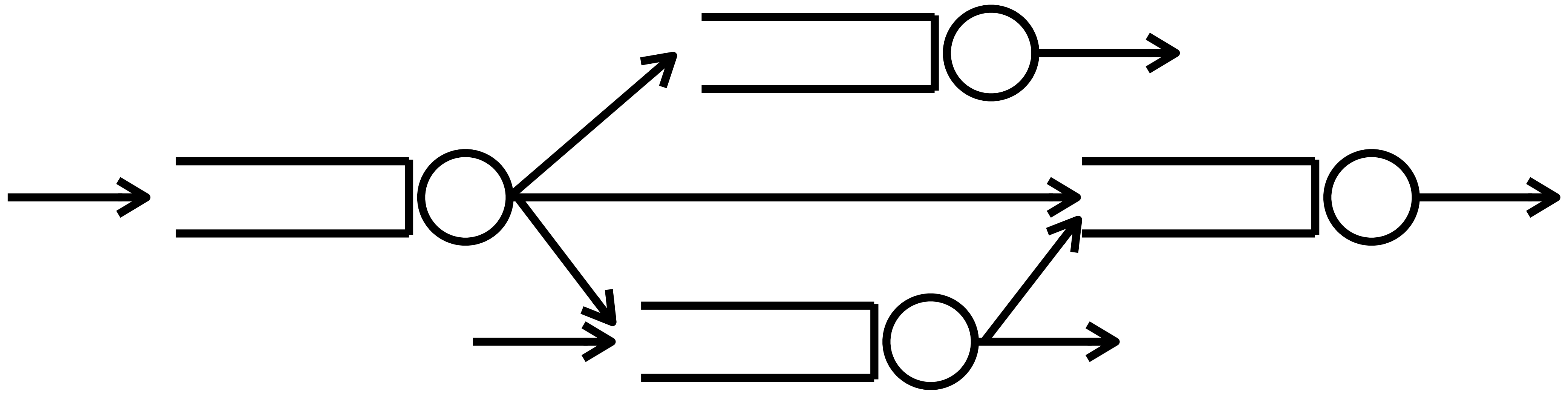

Understanding Delay

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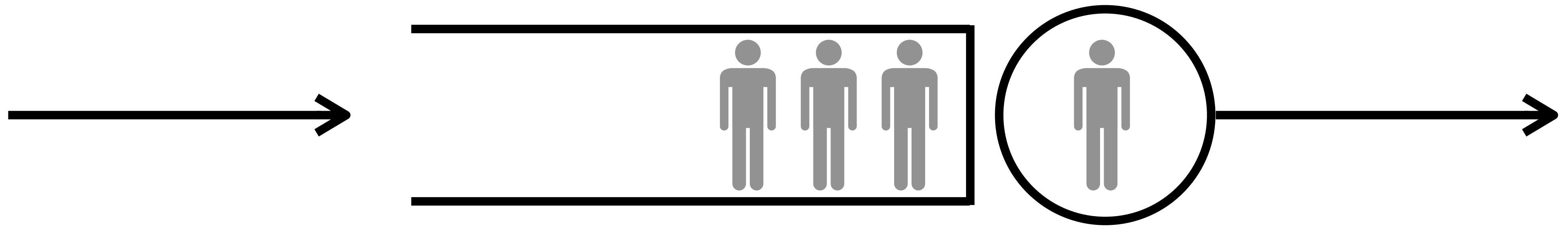

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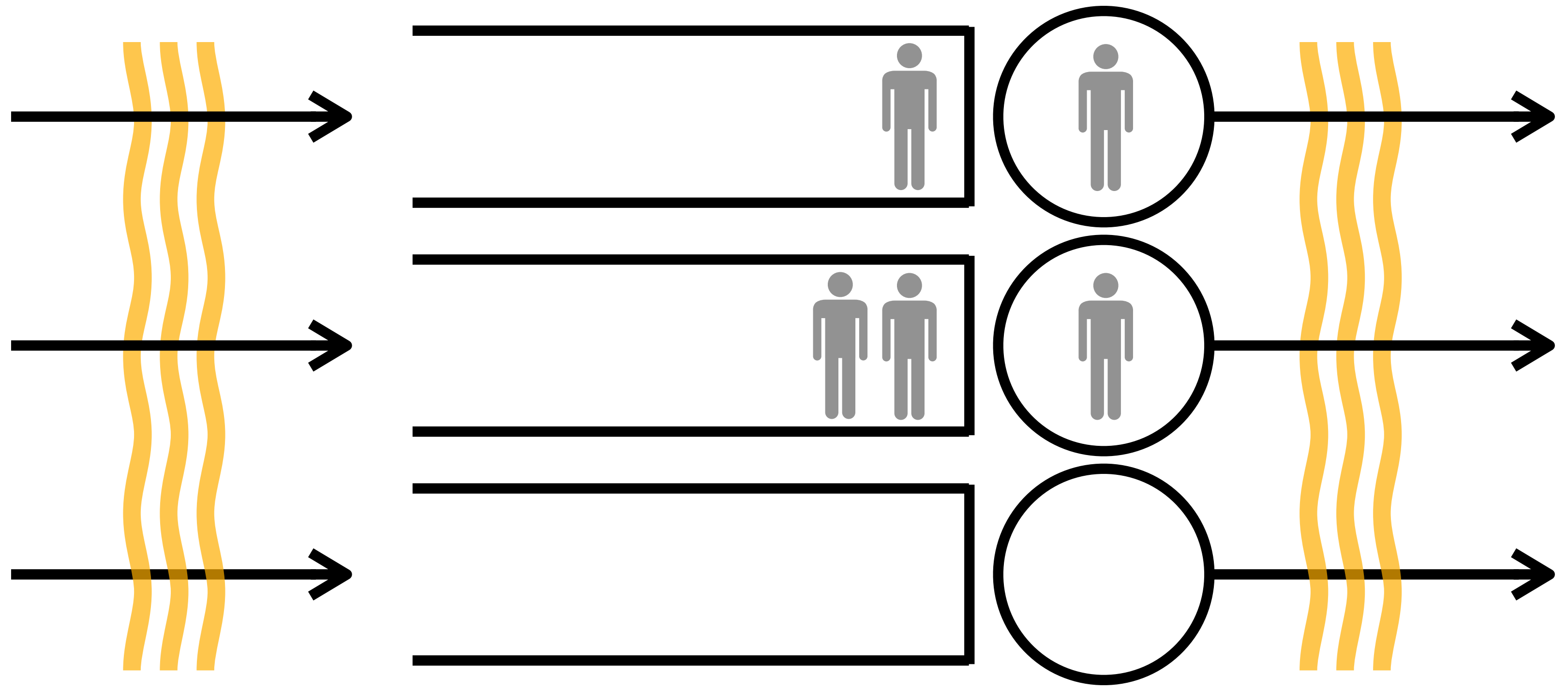

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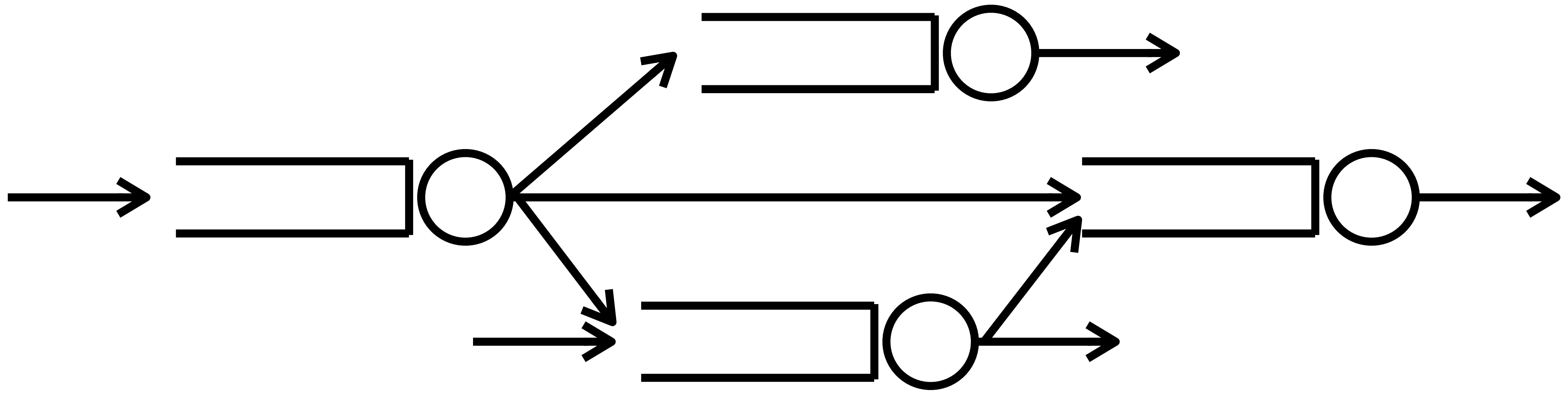
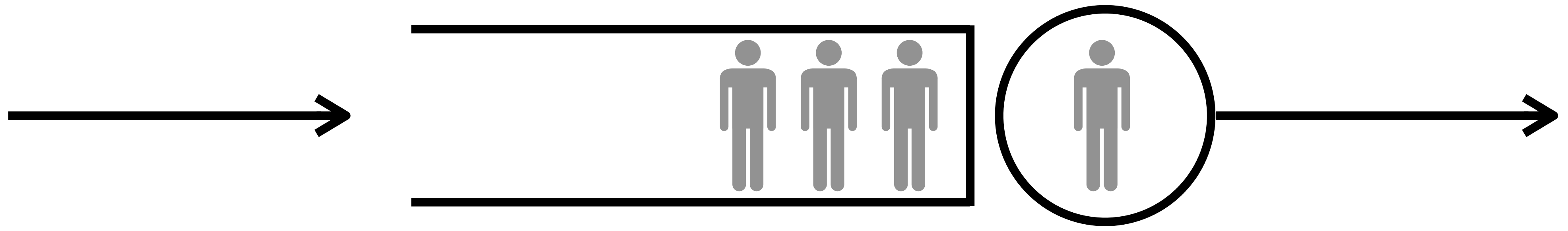


Understanding Delay

Distribution of q?

Mean: $E[q]$?

Single-Server Queue



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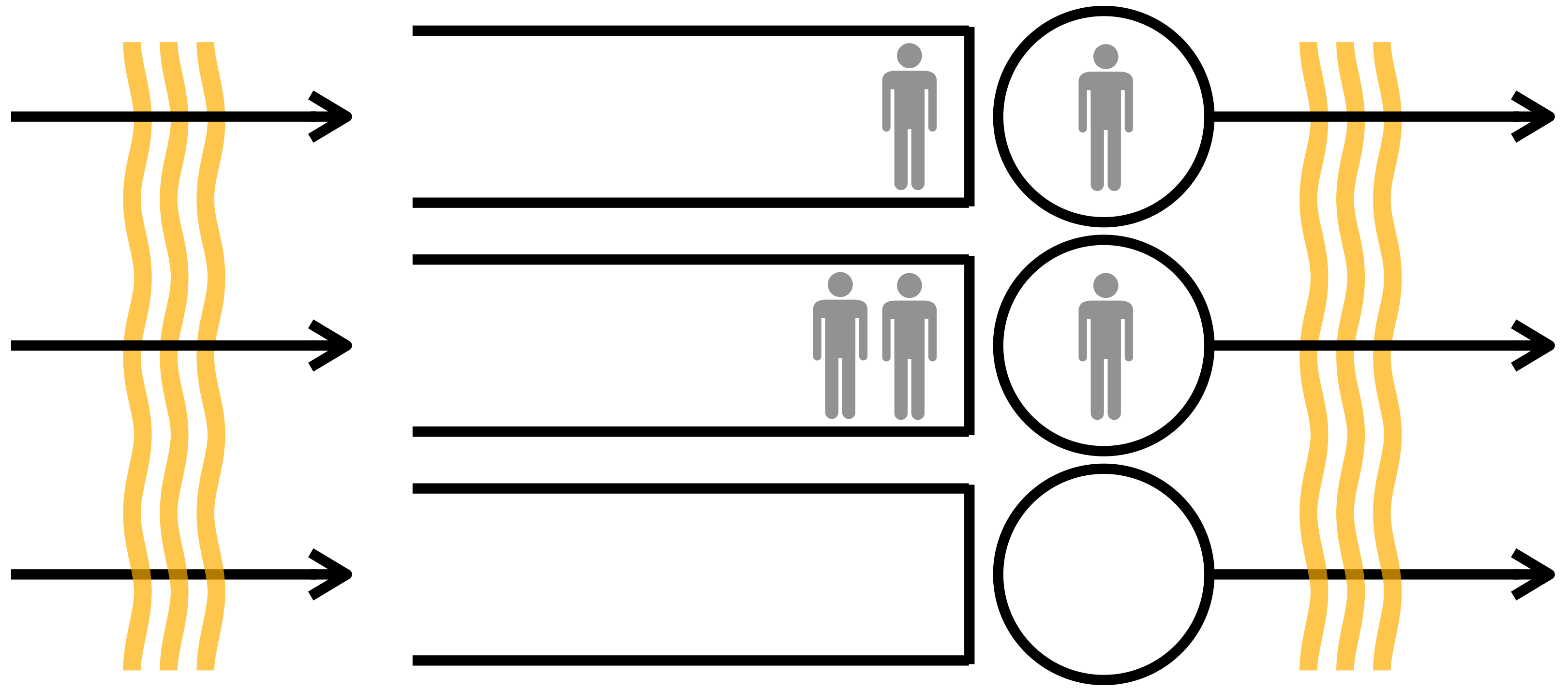

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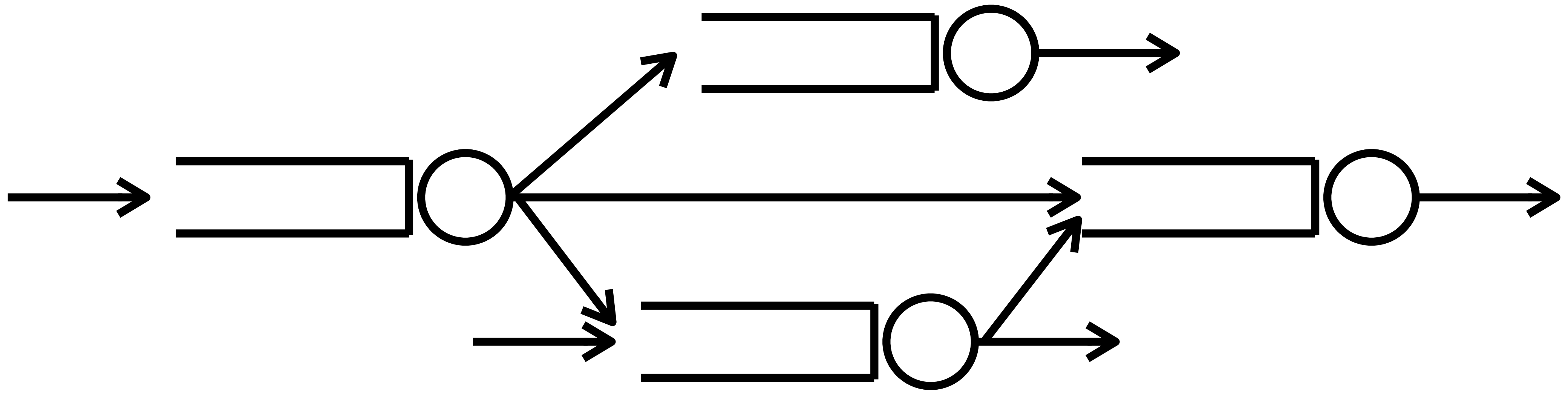

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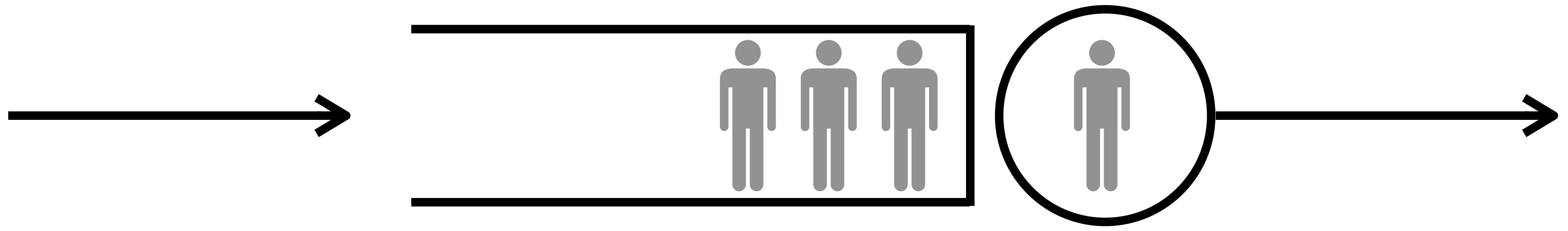

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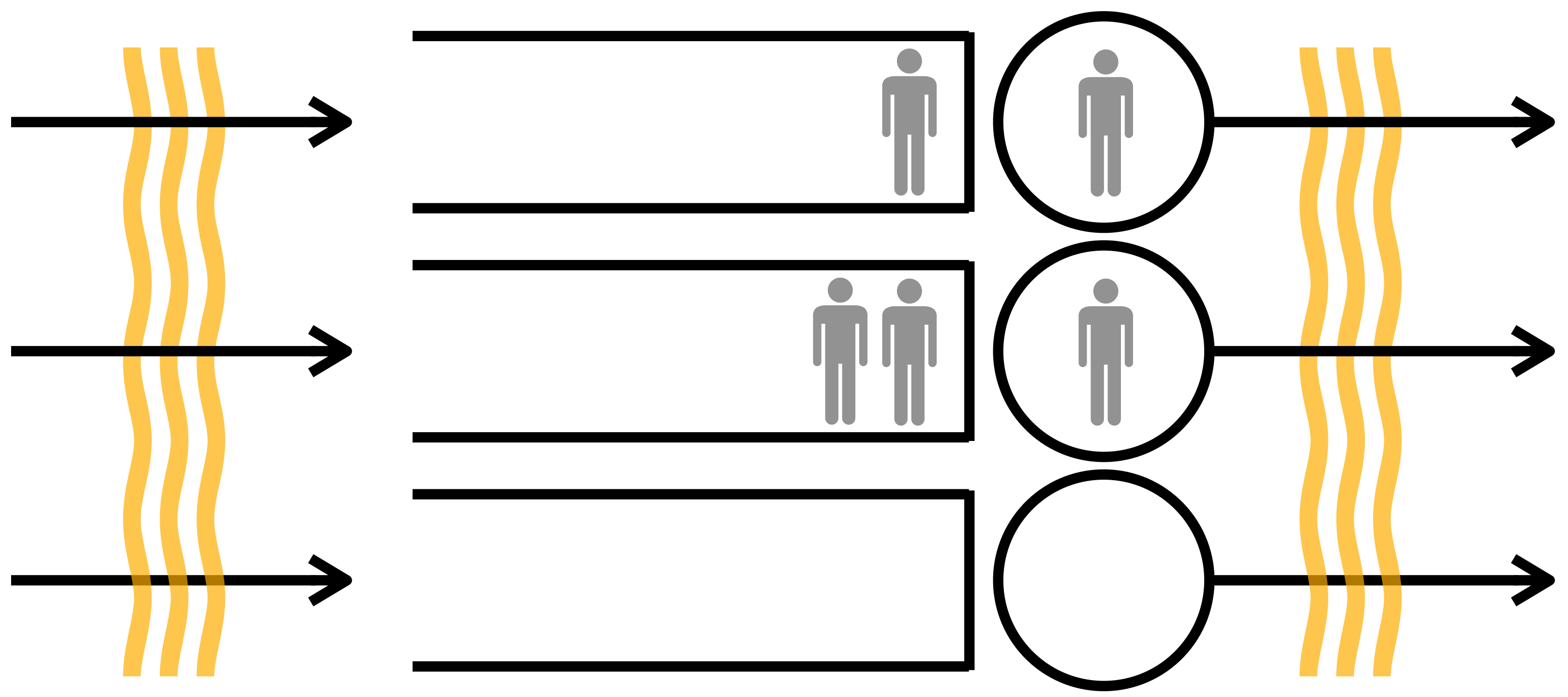

Distribution of $\boldsymbol{q}$?

Mean: $E[\boldsymbol{q}]$? Variance: $Var[\boldsymbol{q}]$?

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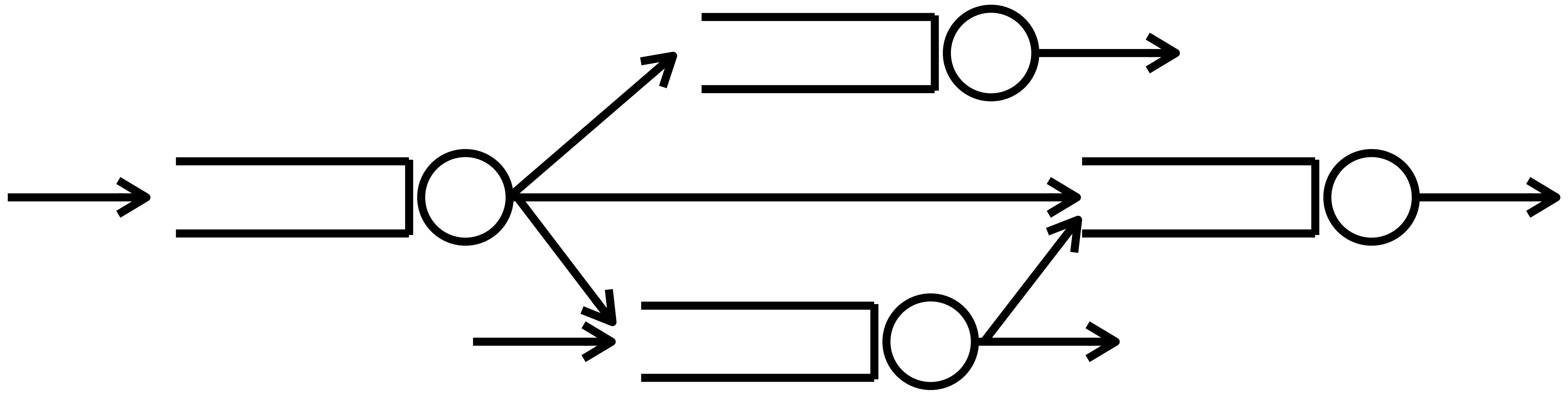

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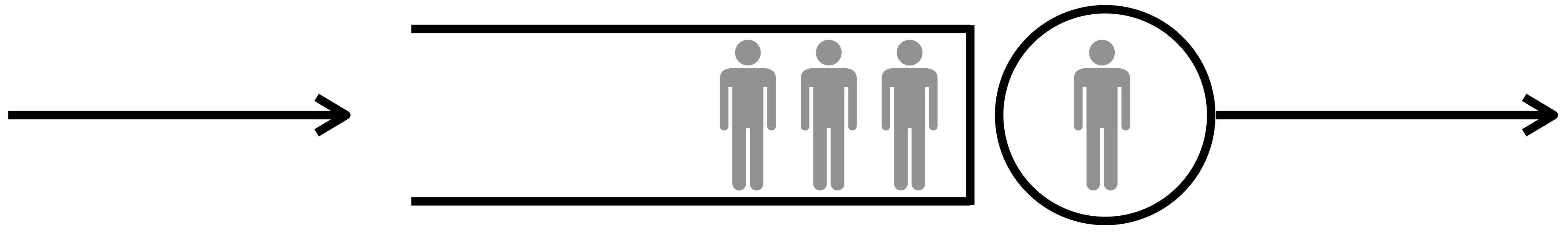

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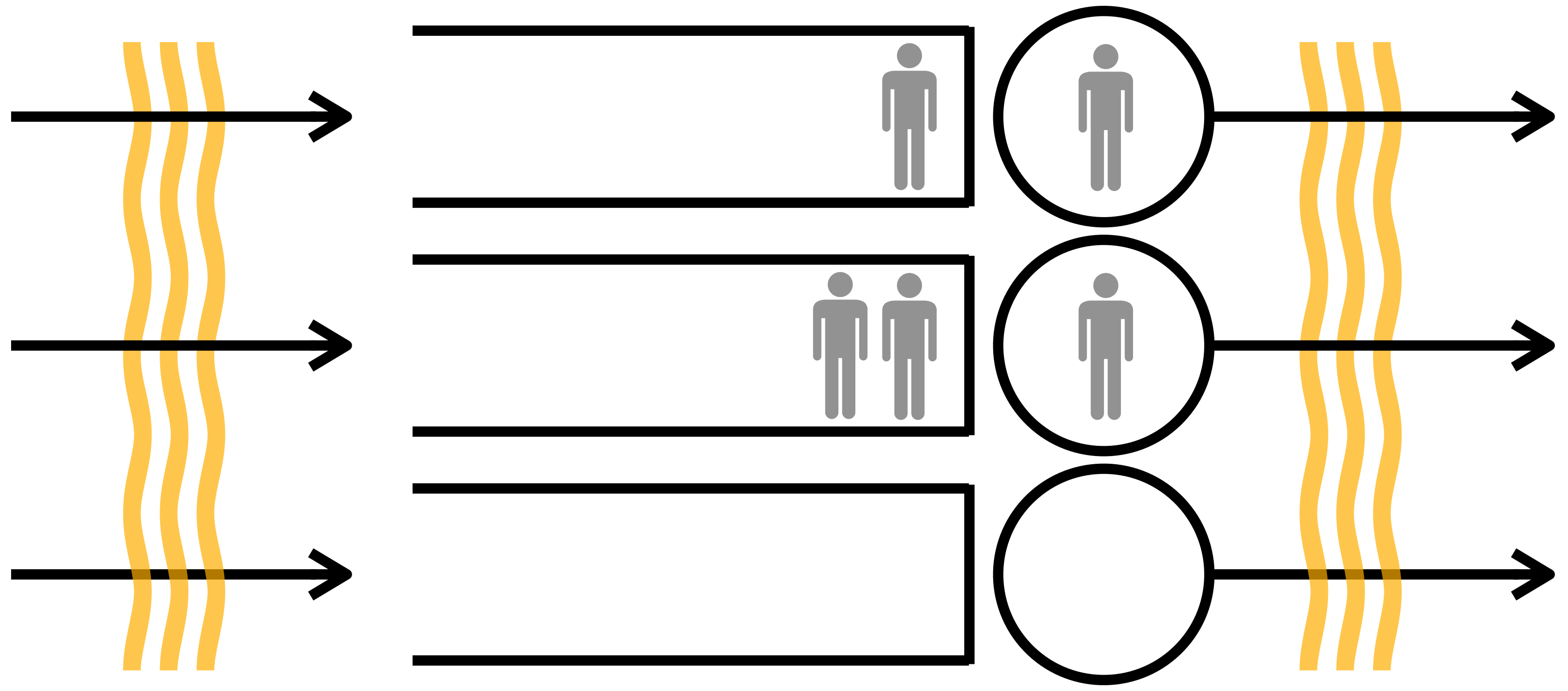

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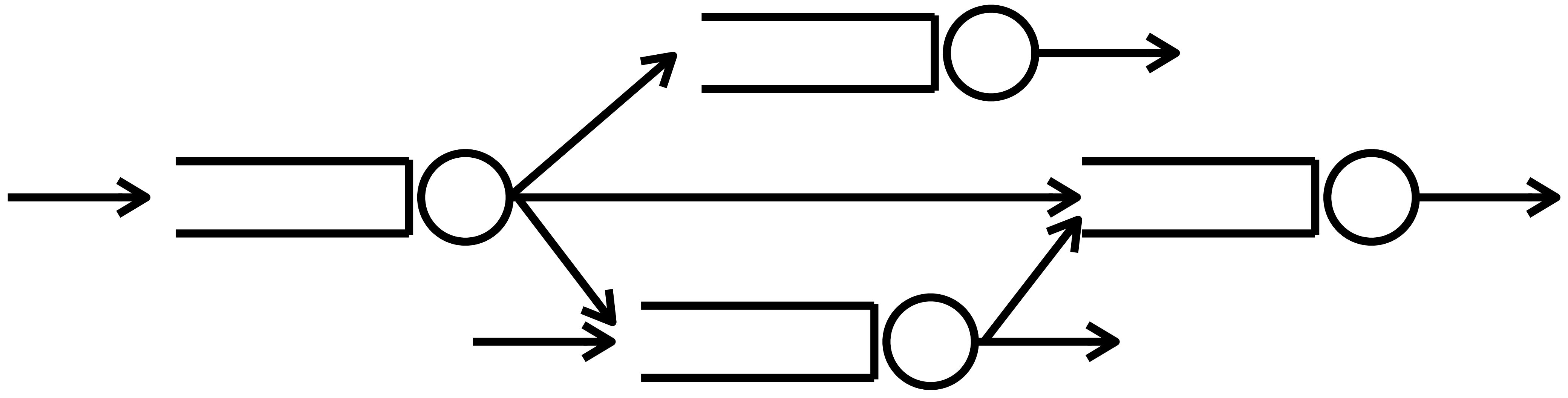

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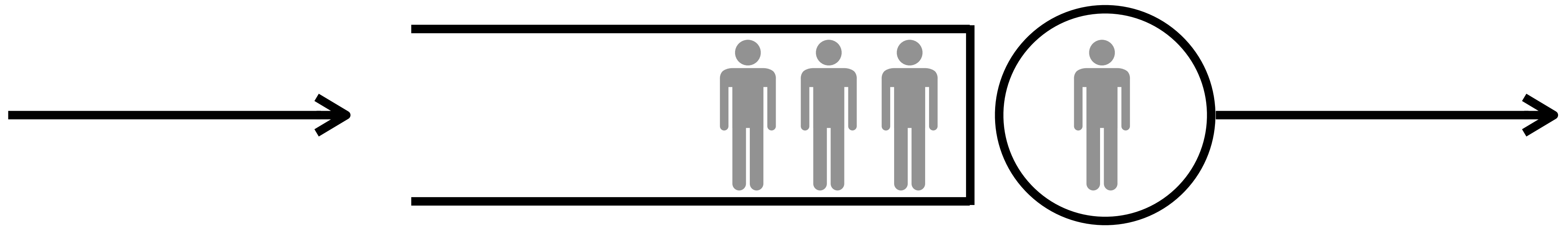

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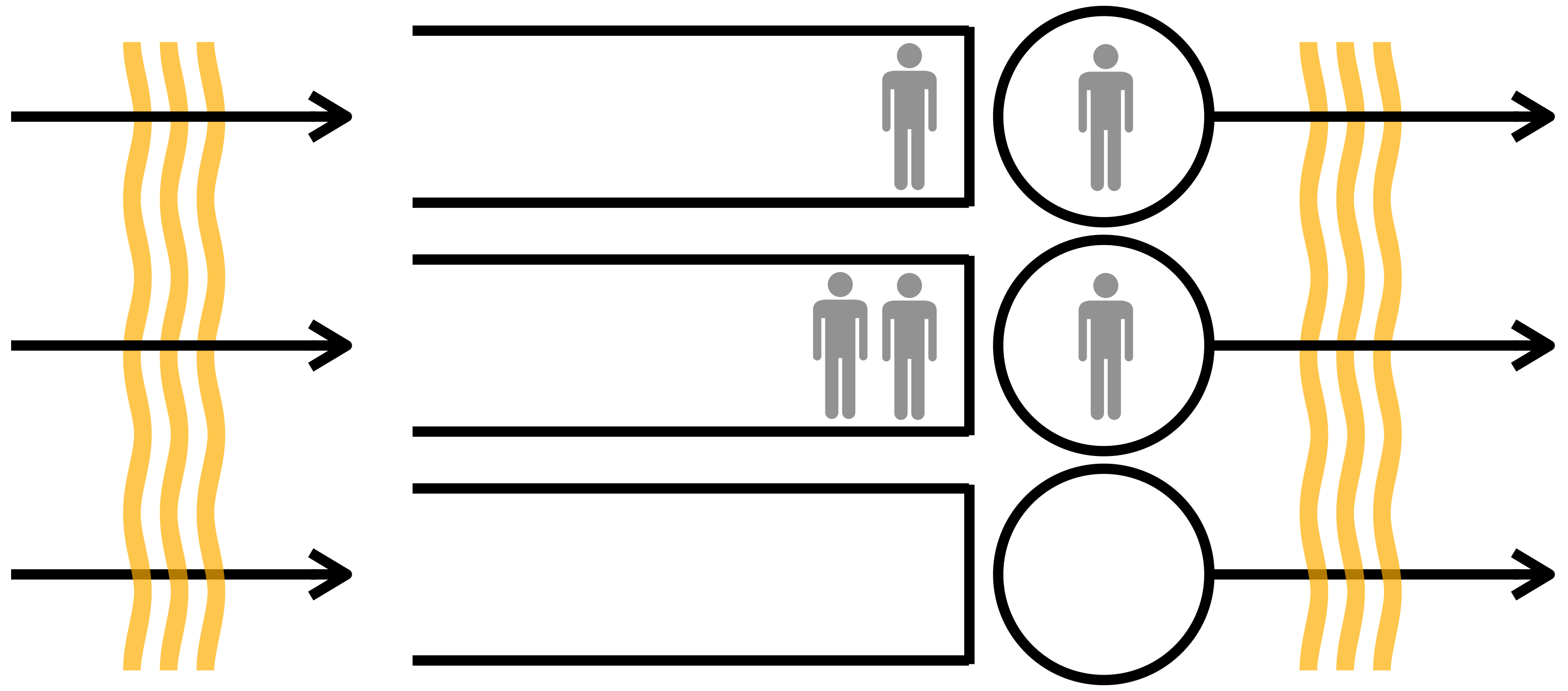

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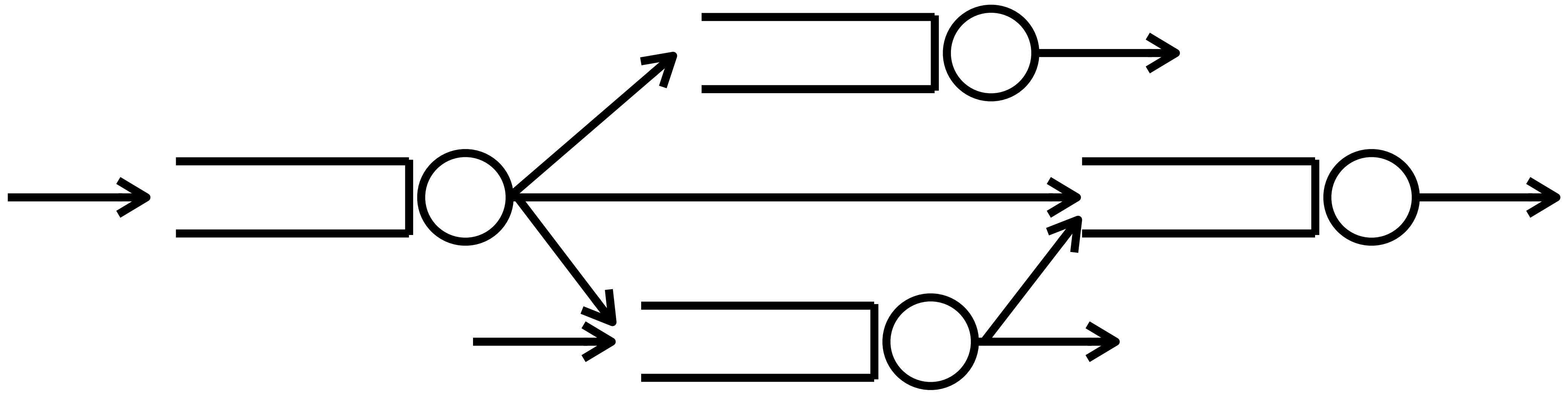

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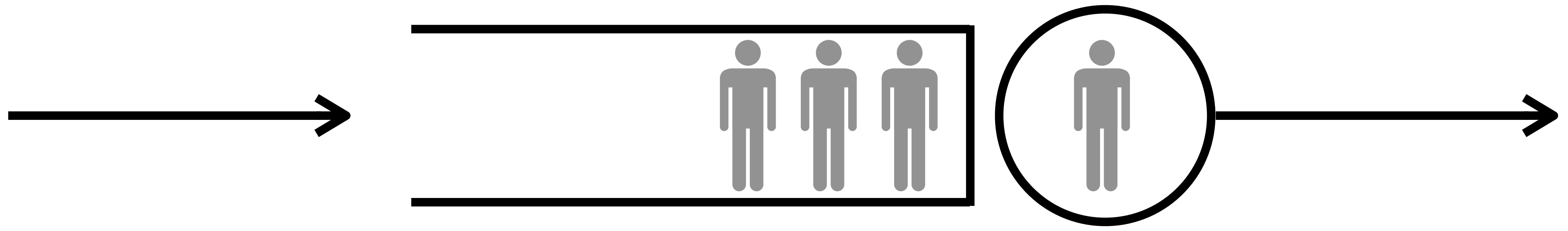

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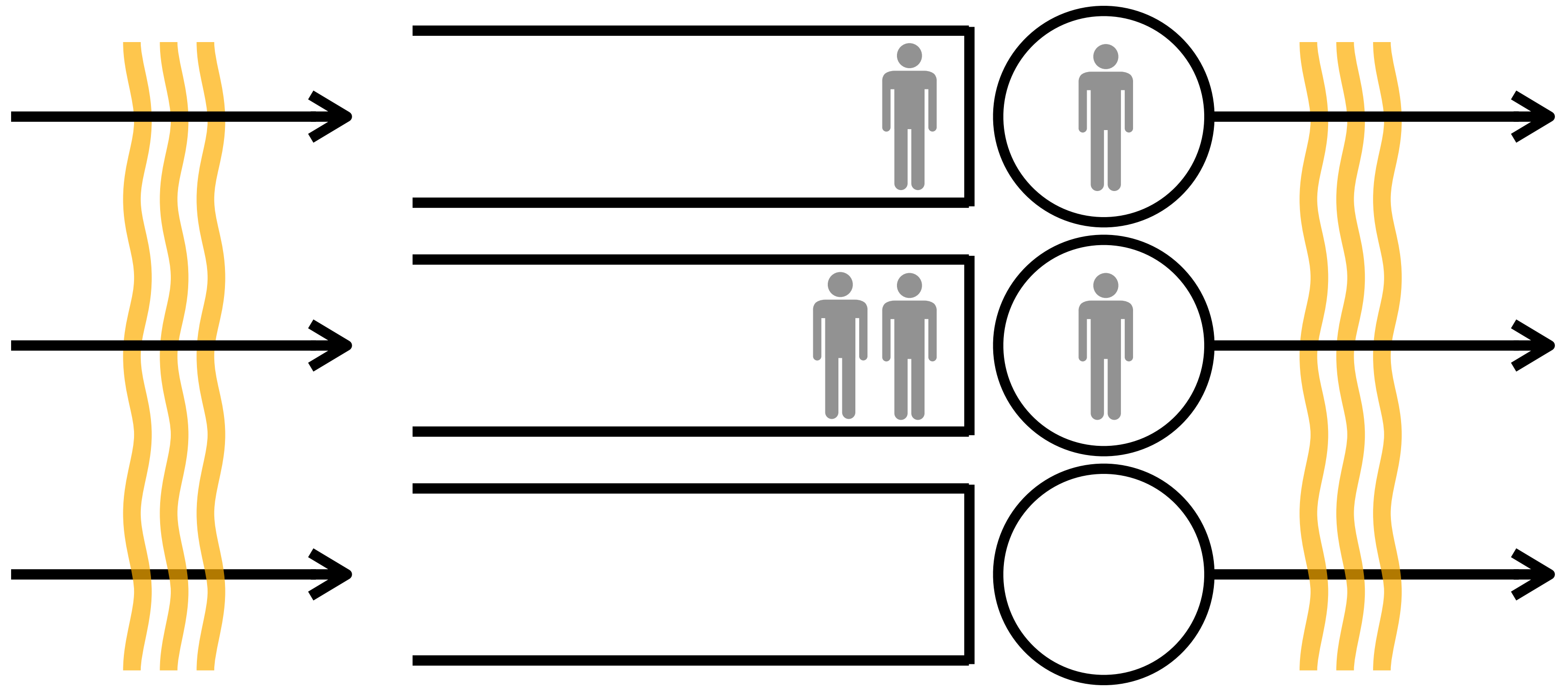

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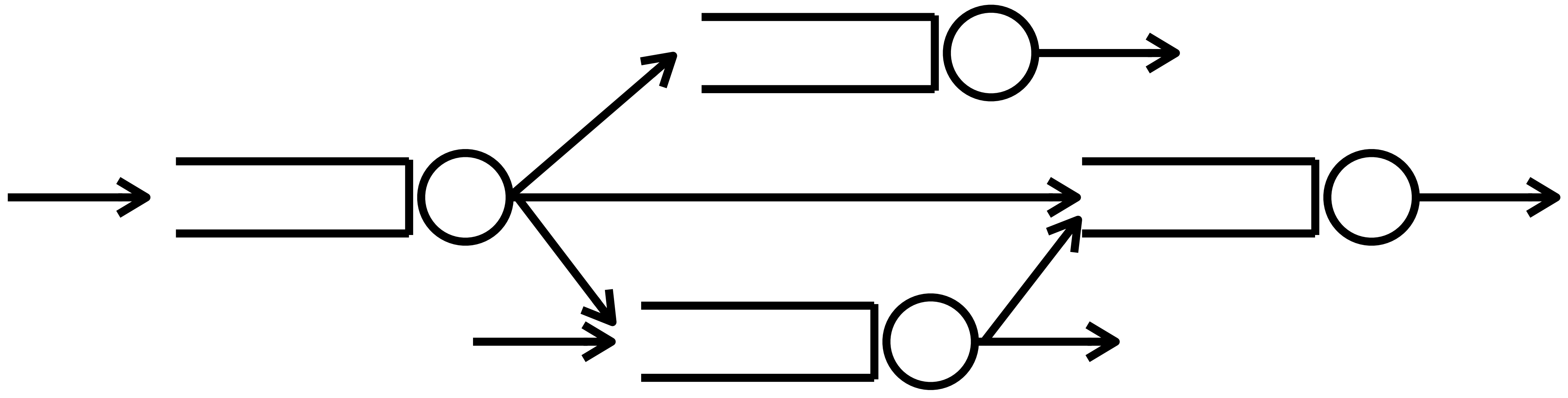

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Distribution is product form

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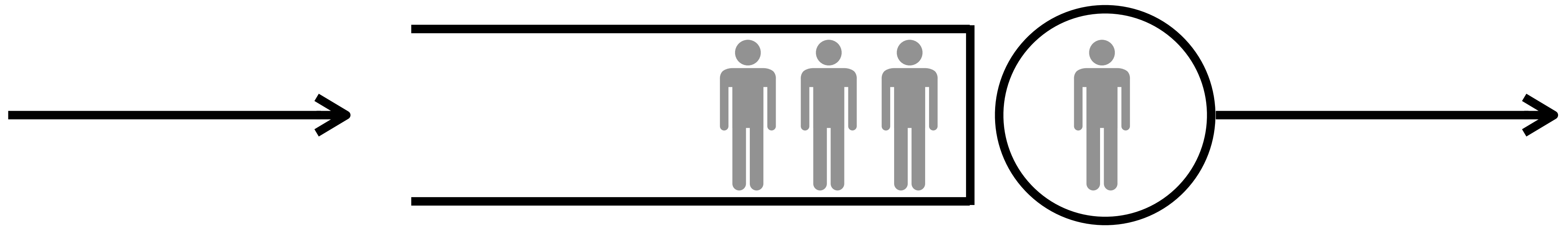

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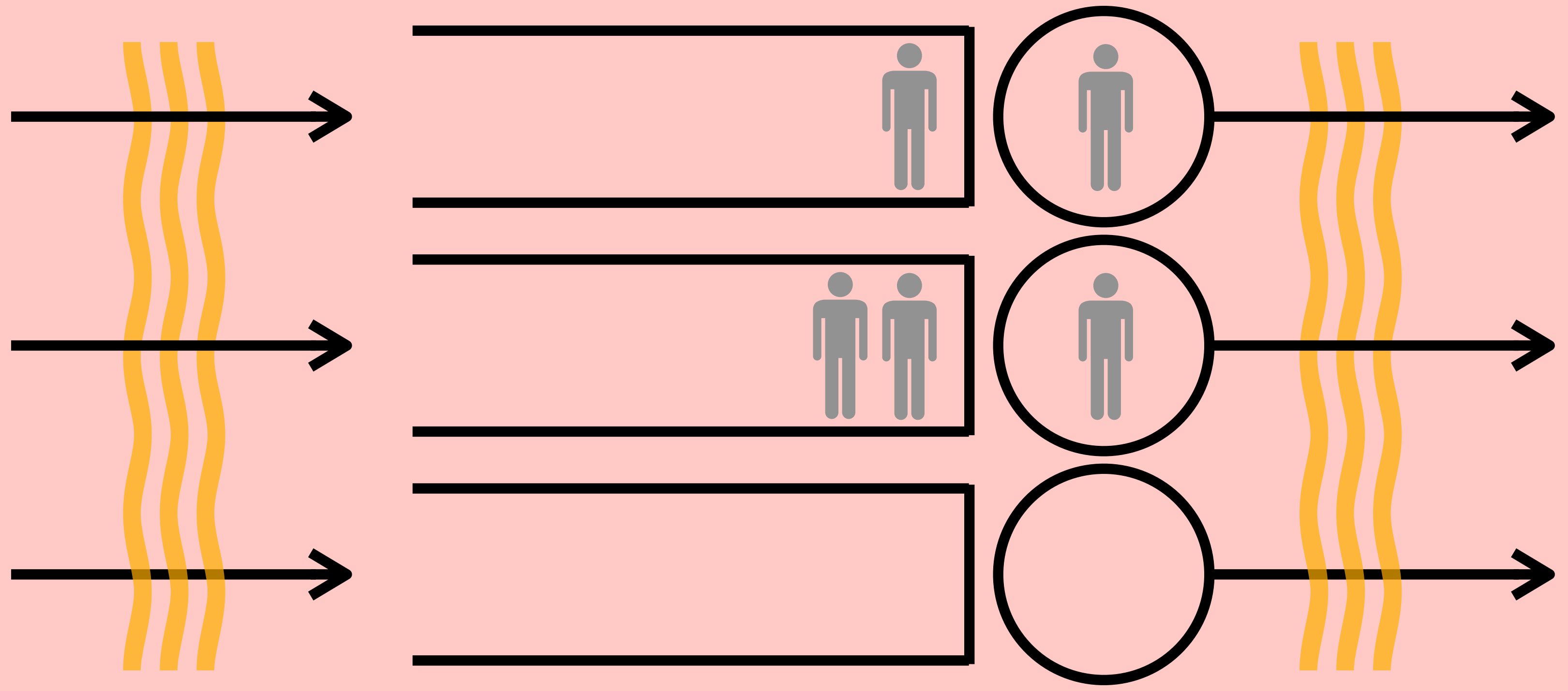
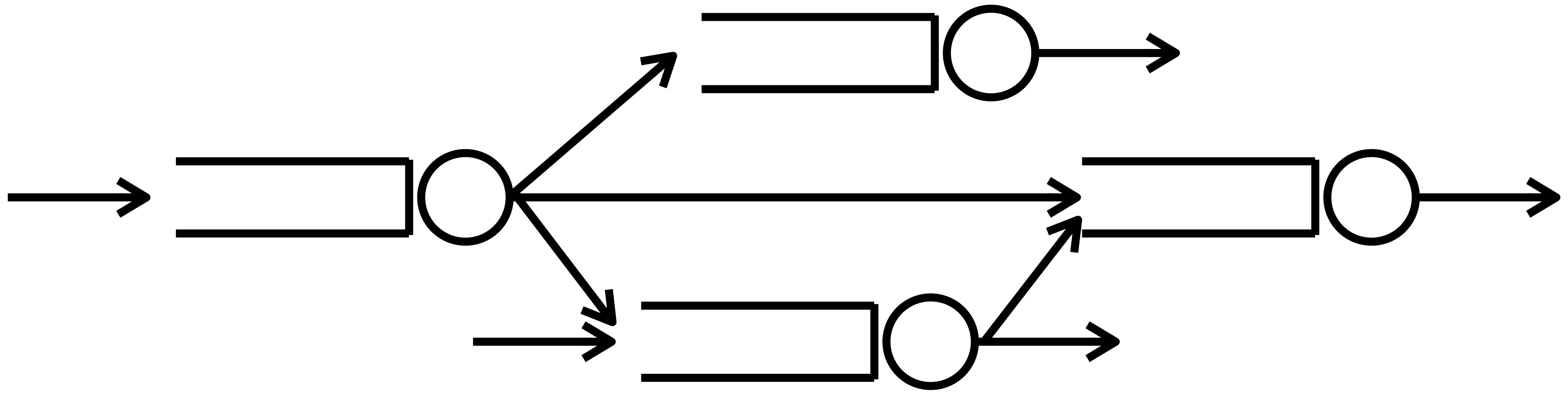


This talk

Networks of queues



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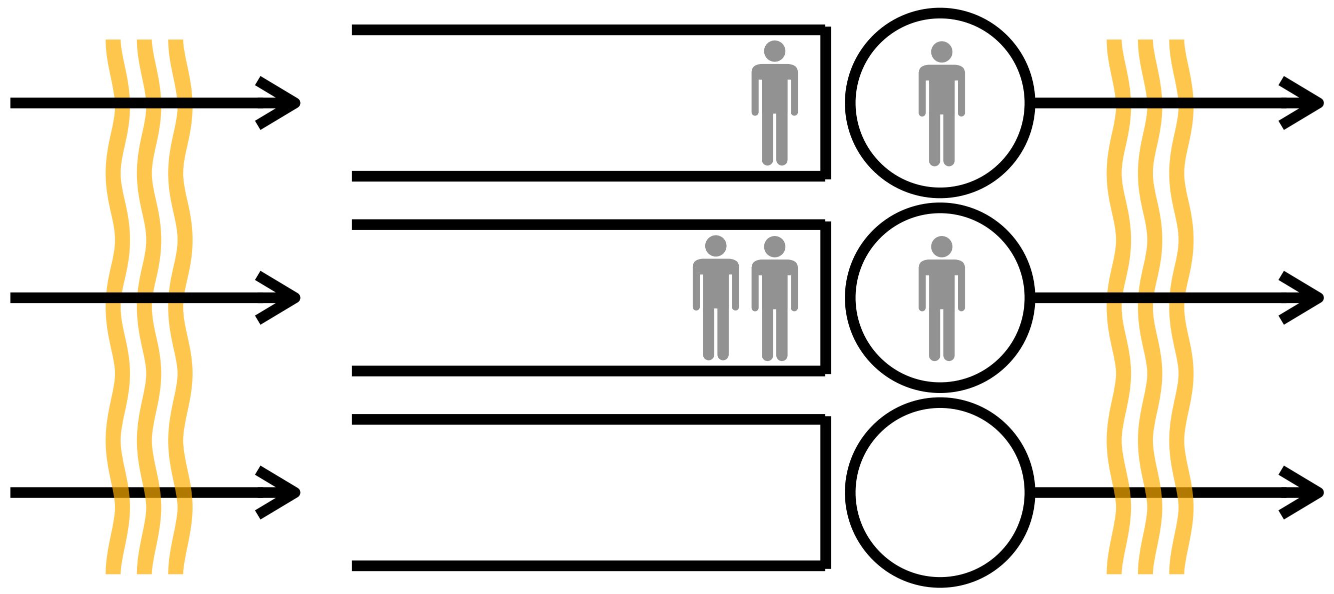
In the literature…

Asymptotic Regimes



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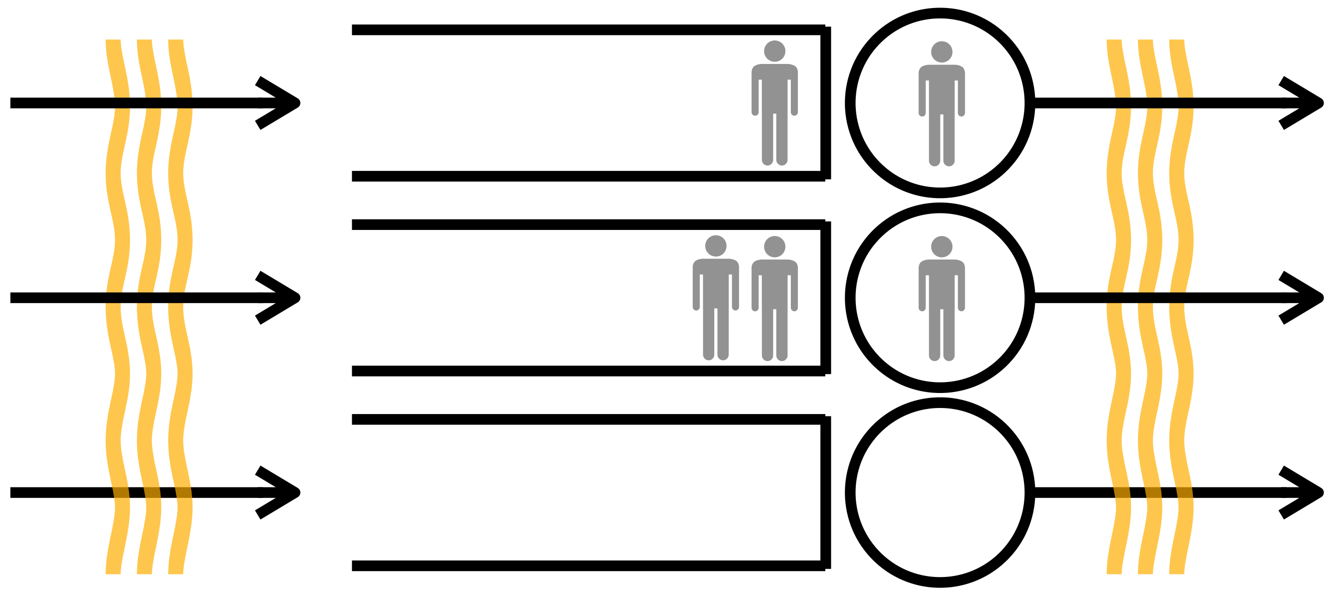

Large Deviations:
(a.k.a. rare events)

$$\lim_{x \to \infty} P(wait > x)$$

Ganesh (2004). Big Queues.
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In the literature…

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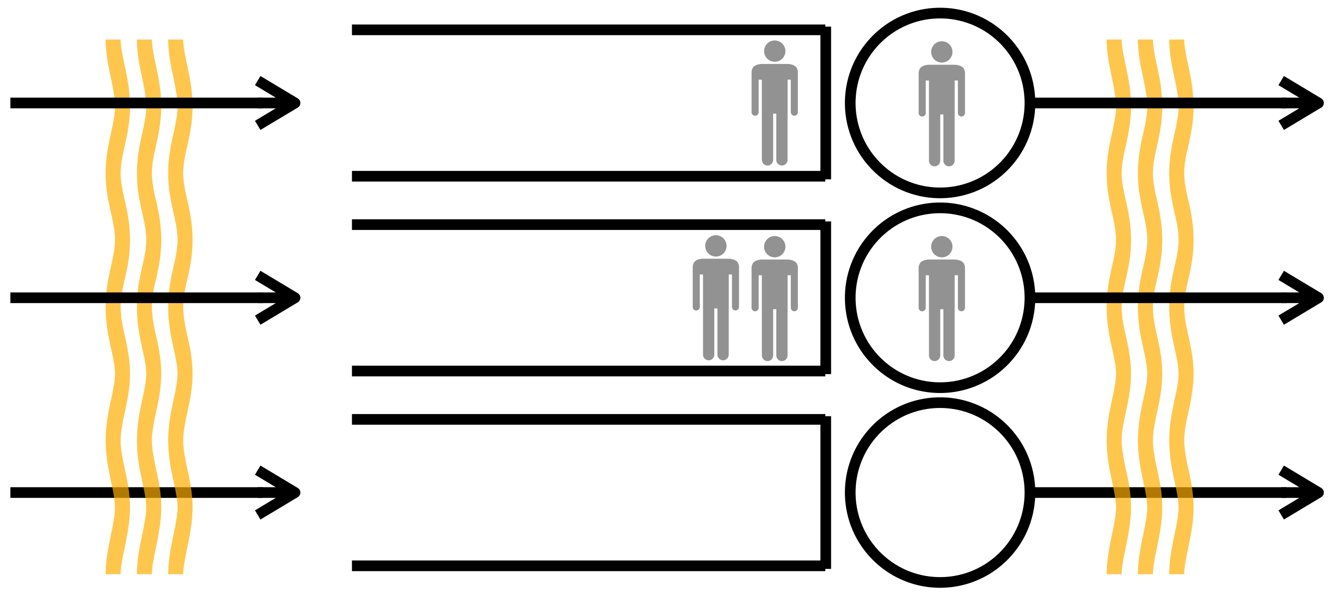

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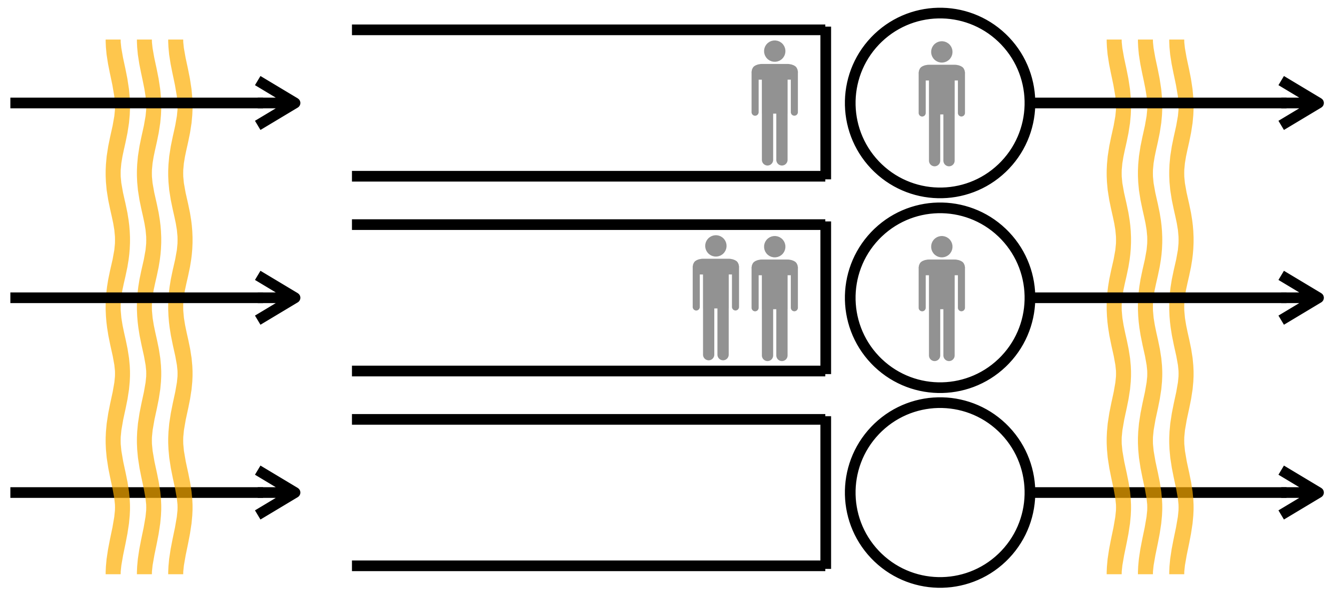

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Heavy Traffic:
Load system to max capacity
$\iff$ Arrival rate $\approx$ Service rate

Kingman (1962)
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Williams (1998, 2000)
Harrison and López (1999)
Stolyar (2004)
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HL et.al. (2022)
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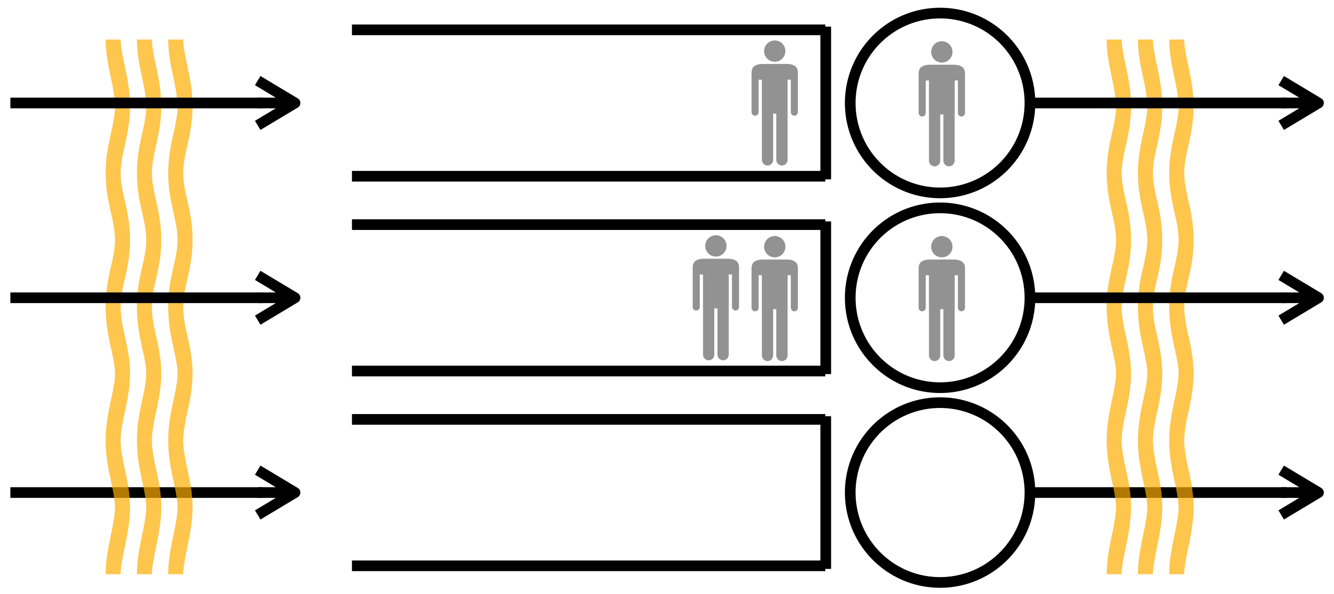

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Many-Server Heavy-Traffic:
\# servers $\to \infty$
& Arrival rate $\approx$ Service rate

Banerjee et.al. (2019, 2020)
Liu and Ying (2019)
Braverman (2020)
Varma et.al. (2022)
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Jhunjhunwala, **HL**, Maguluri (2023)
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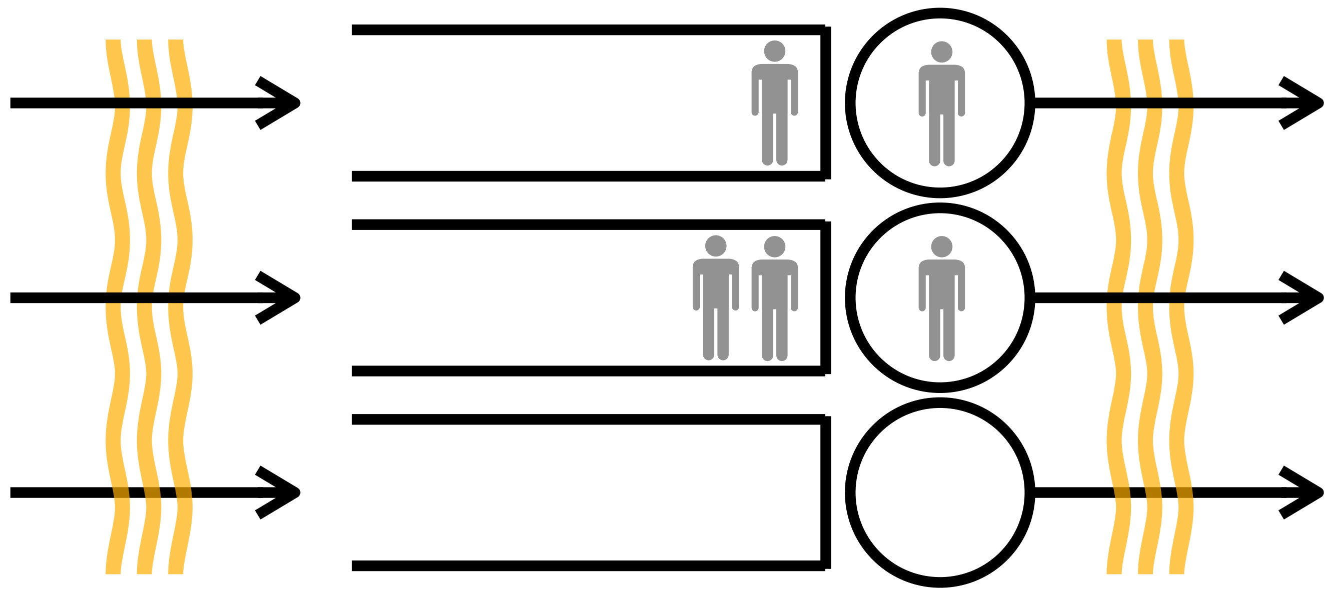

Large Deviations: (a.k.a. rare events) $\lim_{x\to\infty} P(wait > x)$	**Mean Field:** (a.k.a. many servers) # servers $\to \infty$	**Many-Server Heavy-Traffic:** # servers $\to \infty$ & Arrival rate $\approx$ Service rate	**Heavy Traffic:** Load system to max capacity $\iff$ Arrival rate $\approx$ Service rate
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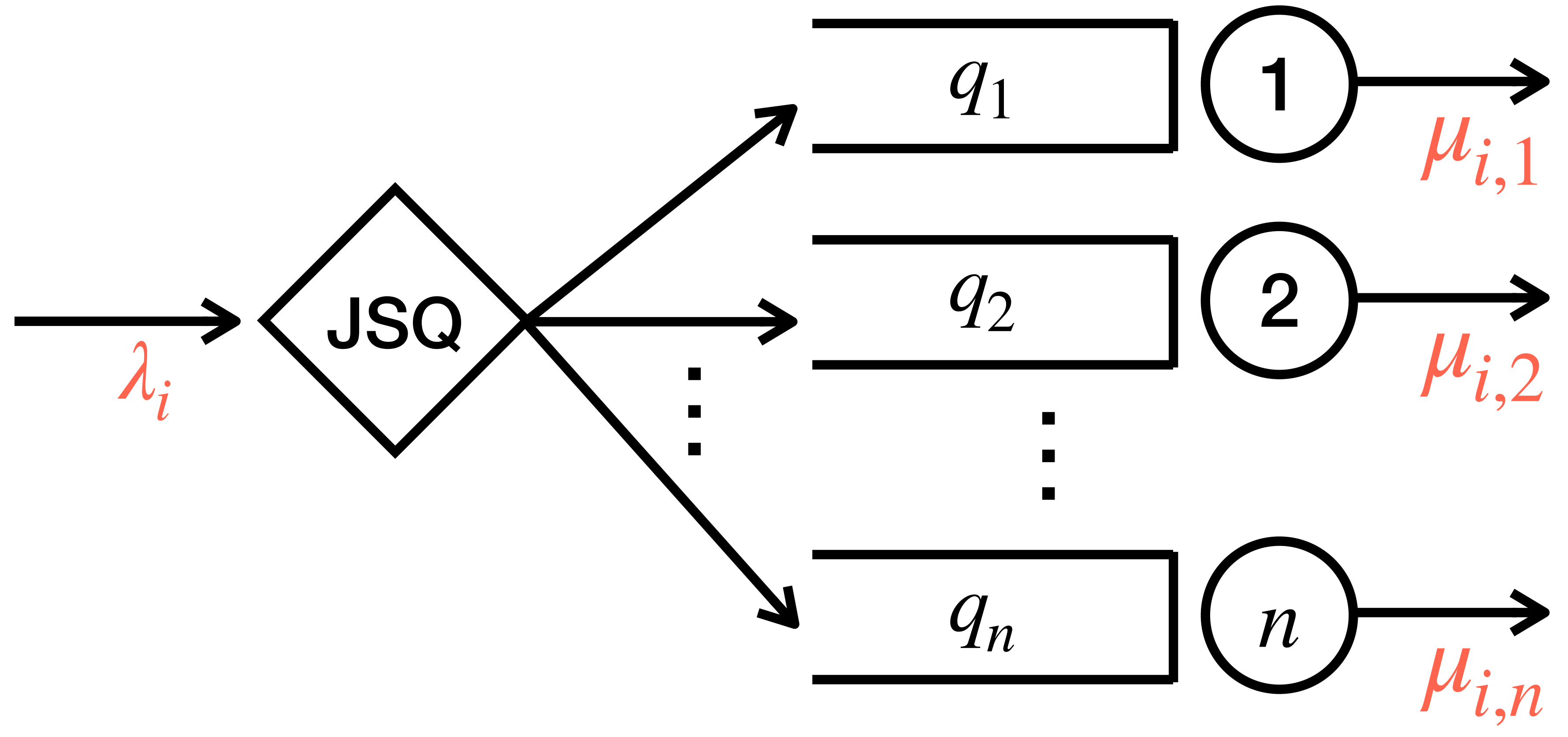
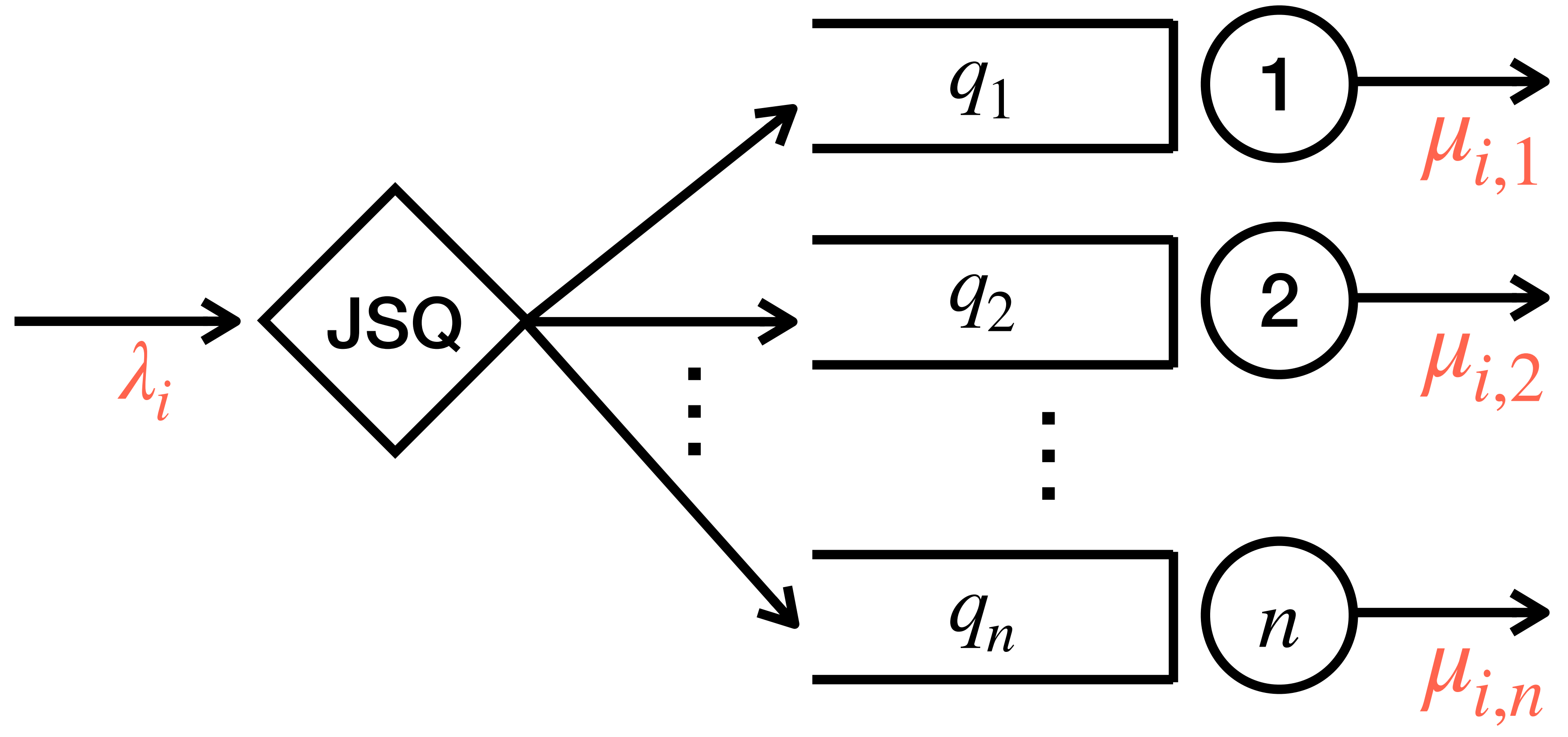
JSQ Model



JSQ Model

- Load balancing under Join-the-Shortest-Queue (JSQ)
- Exponential inter-arrival and service times
- Heterogeneous servers



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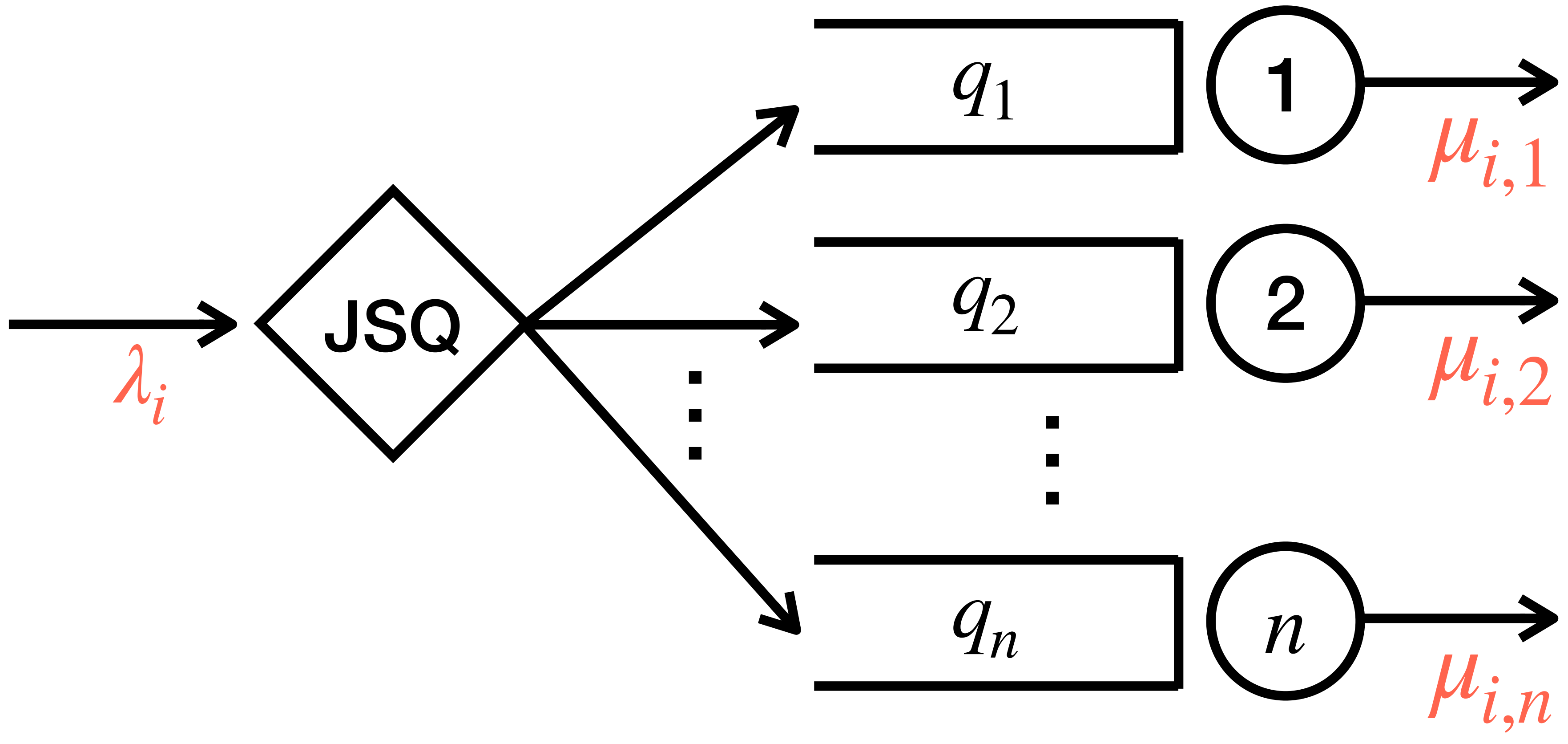
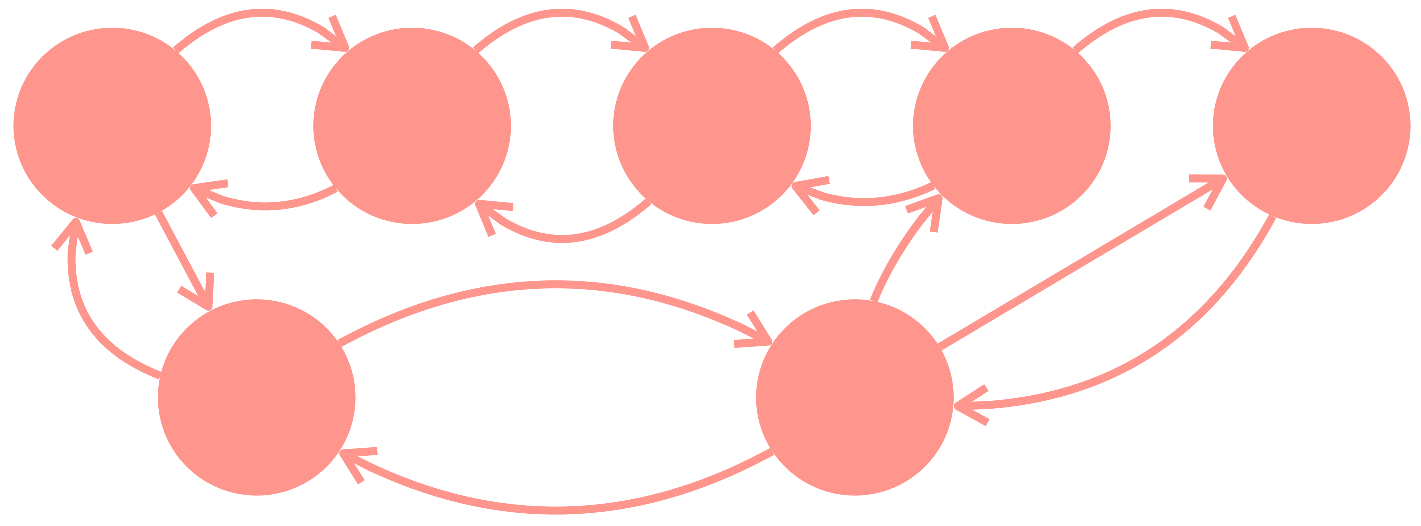


$Z(t) = i \sim$ Markov chain



$$\rho_\pi = \frac{\mathbb{E}[\lambda_i \,|\, i \sim \pi]}{\sum_j \mathbb{E}[\mu_{ij} \,|\, i \sim \pi]}, \quad \epsilon = 1 - \rho_\pi$$

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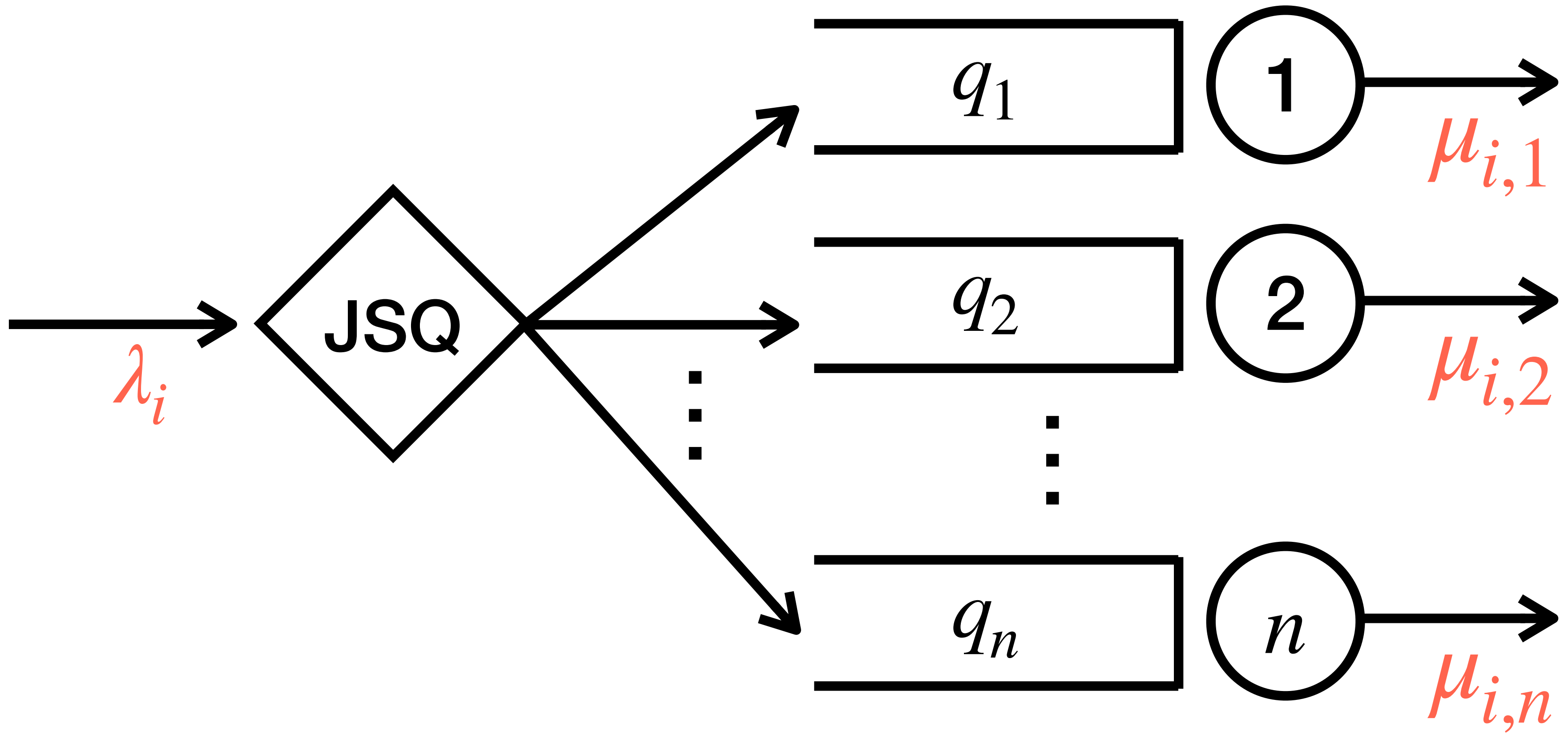
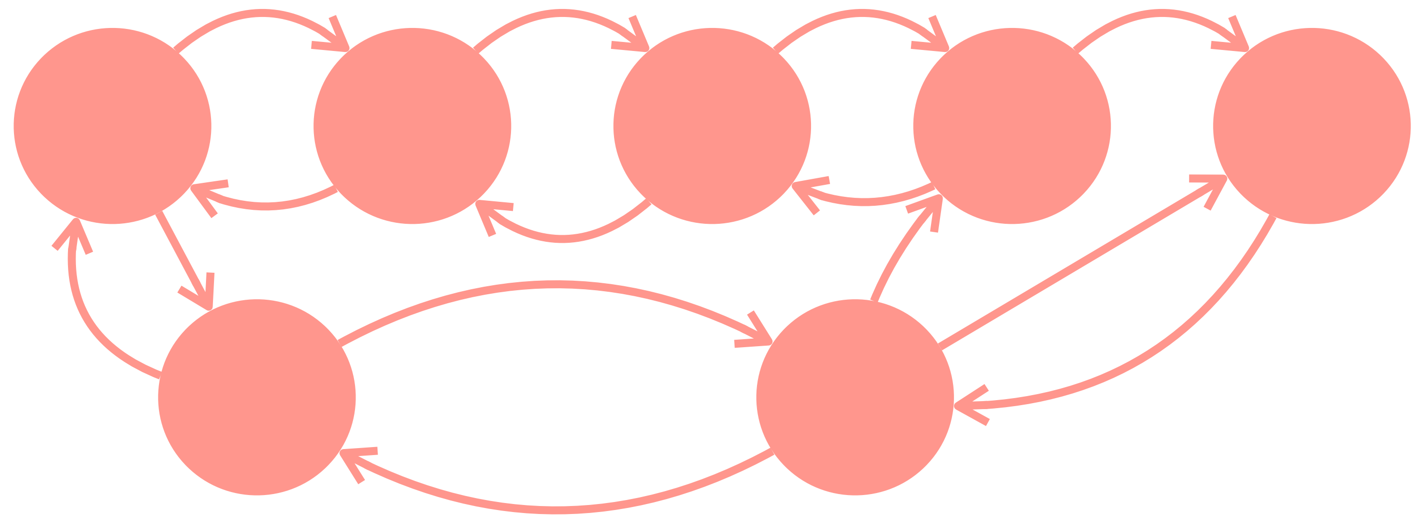


Assumptions on $\{Z(t)\}_t$

- Countable state space
- Stationary distribution exists
- $\lambda_{\max} < \infty$ and $\mu_{\max} < \infty$

$Z(t) = i \sim$ Markov chain



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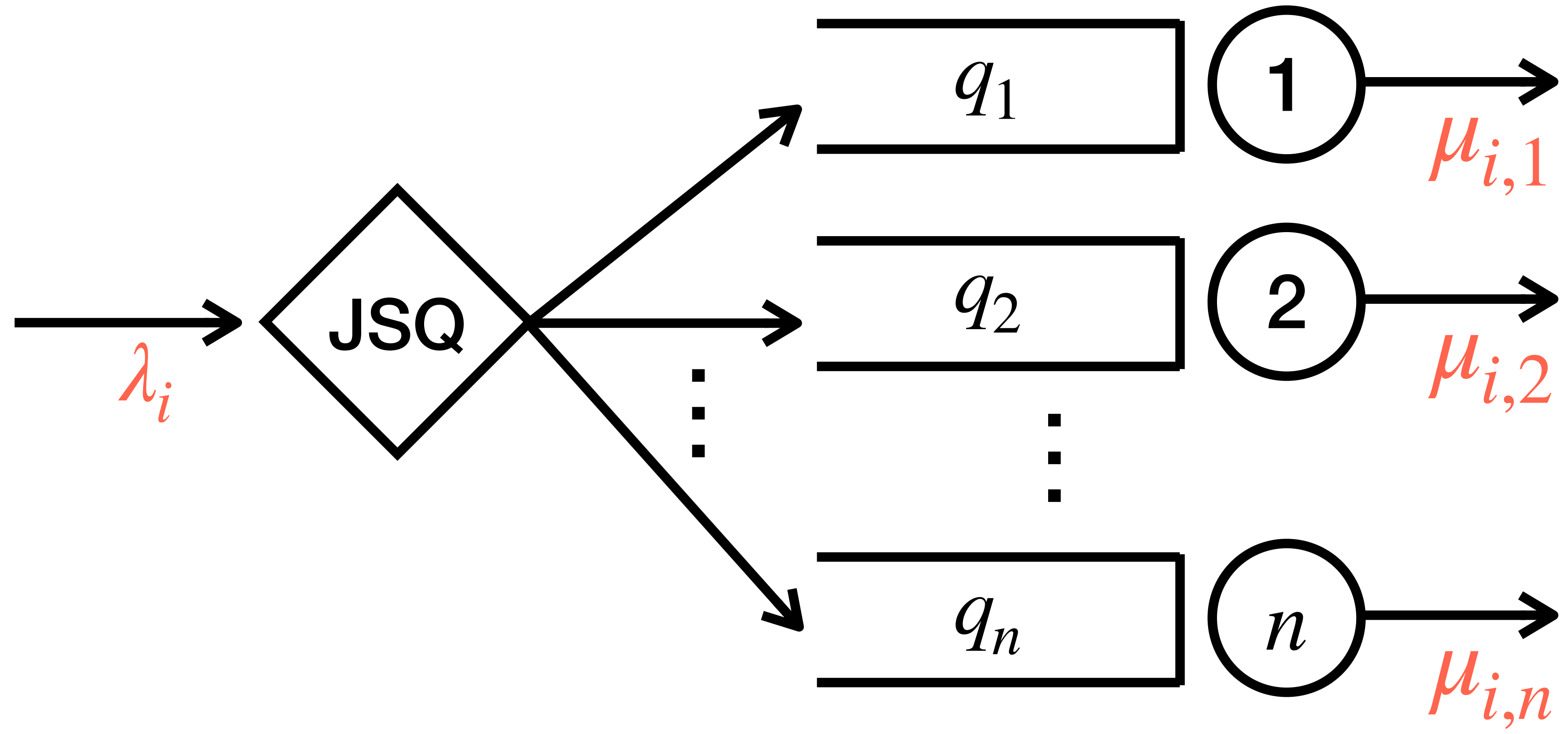
Asymptotic Distribution of Queue Lengths



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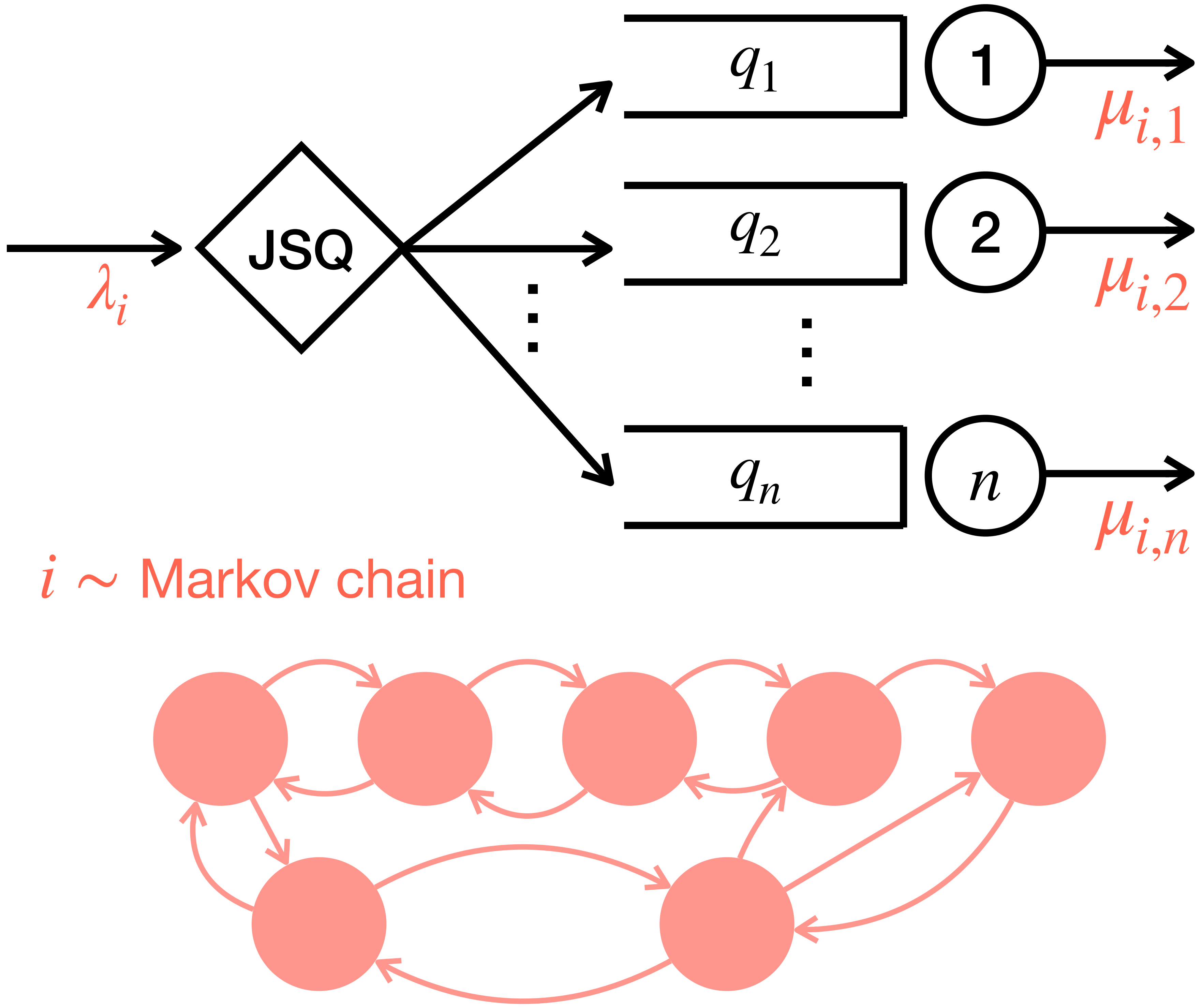

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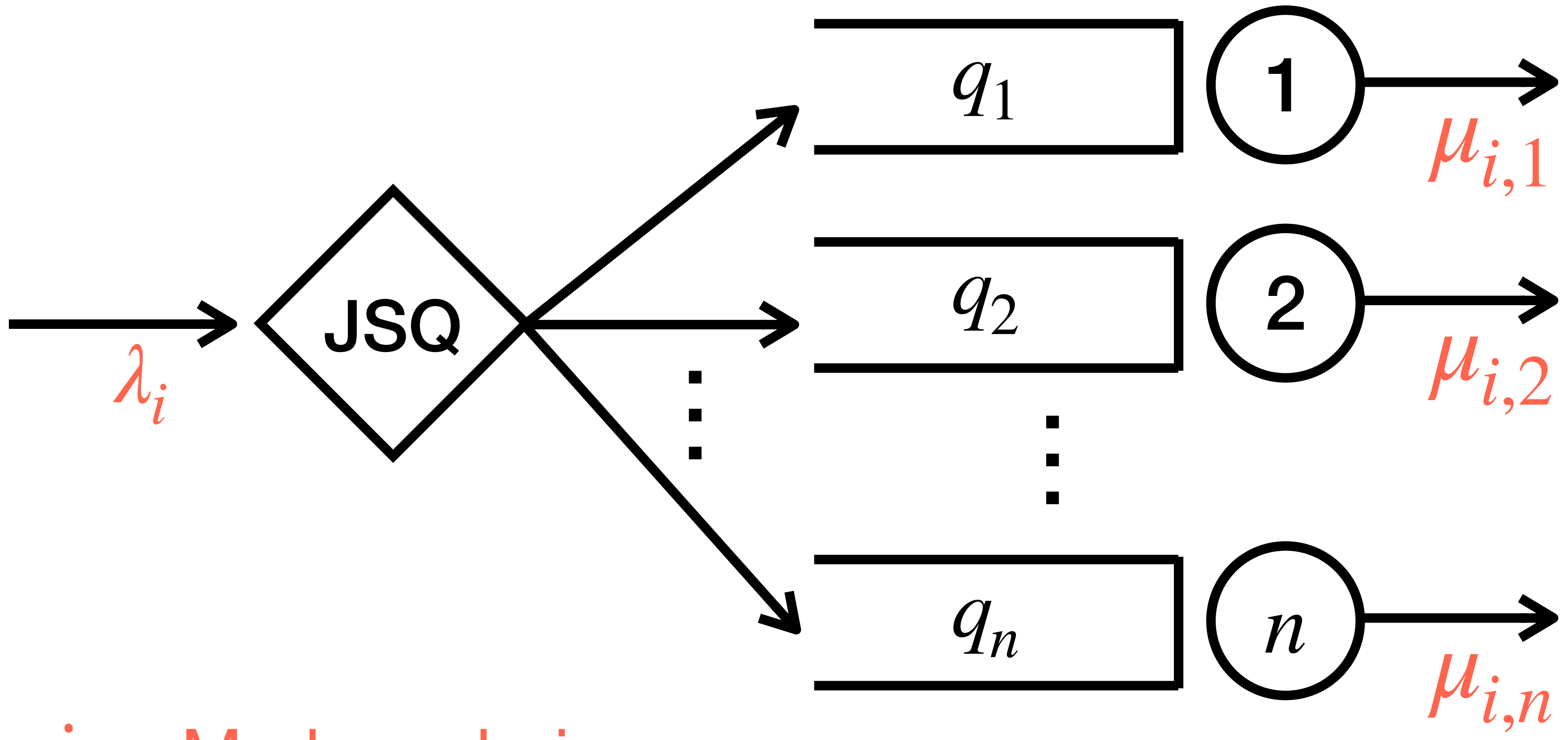
Asymptotic Distribution of Queue Lengths

Theorem [HL, Grosof '25]**:**

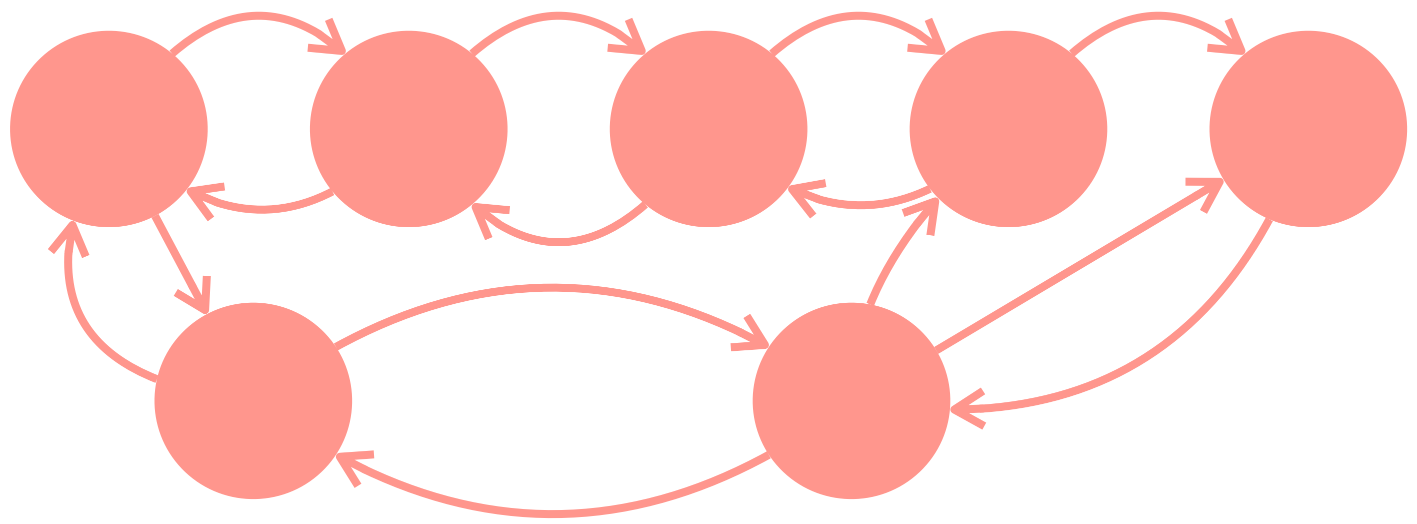
If $\lambda_i > 0$ is large enough for each i, then as $\epsilon \downarrow 0$

$$\epsilon \boldsymbol{q} \Longrightarrow Exp\left(\text{Mean} = 1 + \frac{k^*}{\mu_\Sigma}\right)$$

≈ Variance of arrival and service times in steady state



$i \sim$ Markov chain



$$\rho_\pi = \frac{\mathbb{E}[\lambda_i \,|\, i \sim \pi]}{\sum_j \mathbb{E}[\mu_{ij} \,|\, i \sim \pi]}, \quad \epsilon = 1 - \rho_\pi$$

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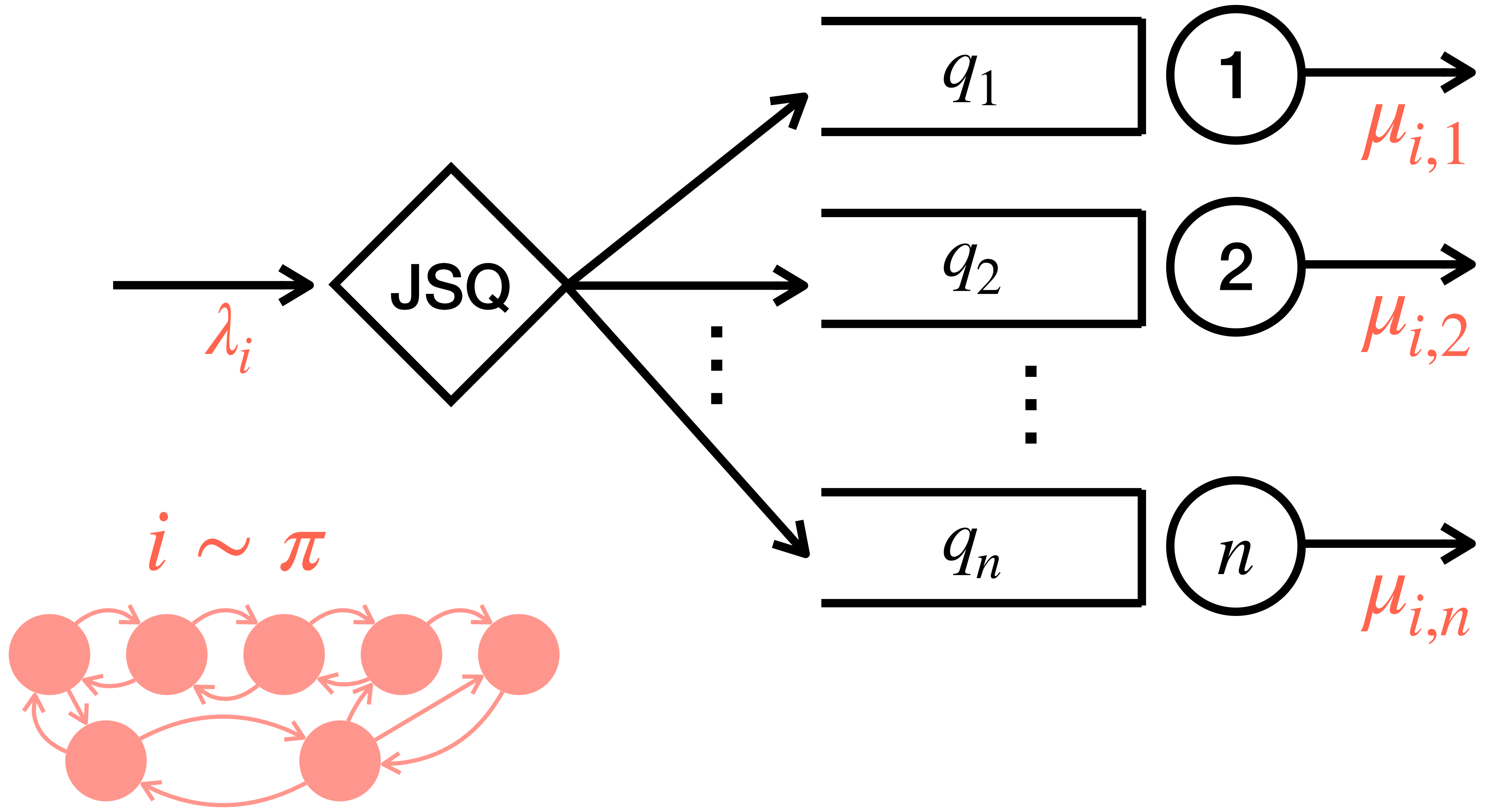

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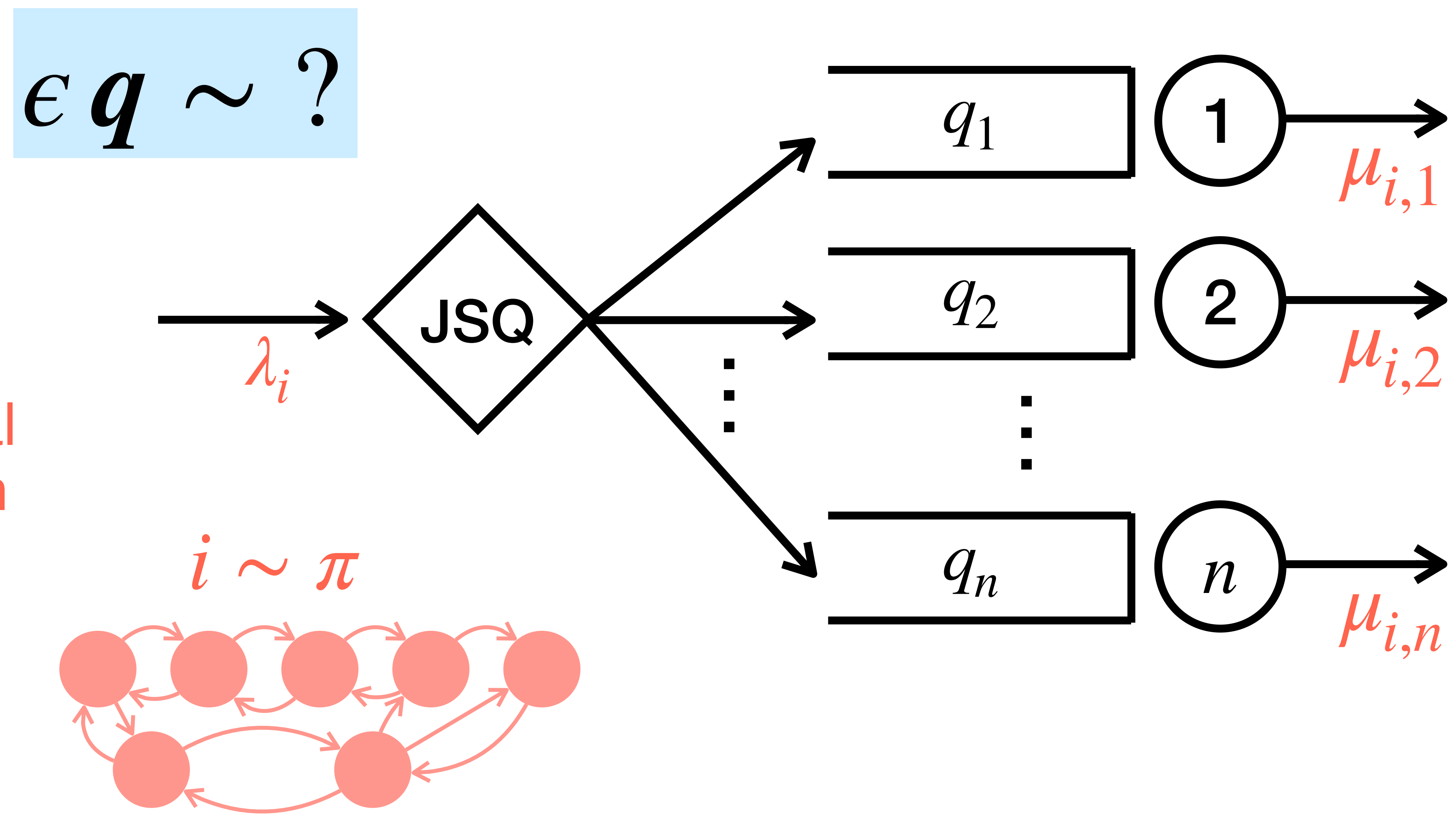

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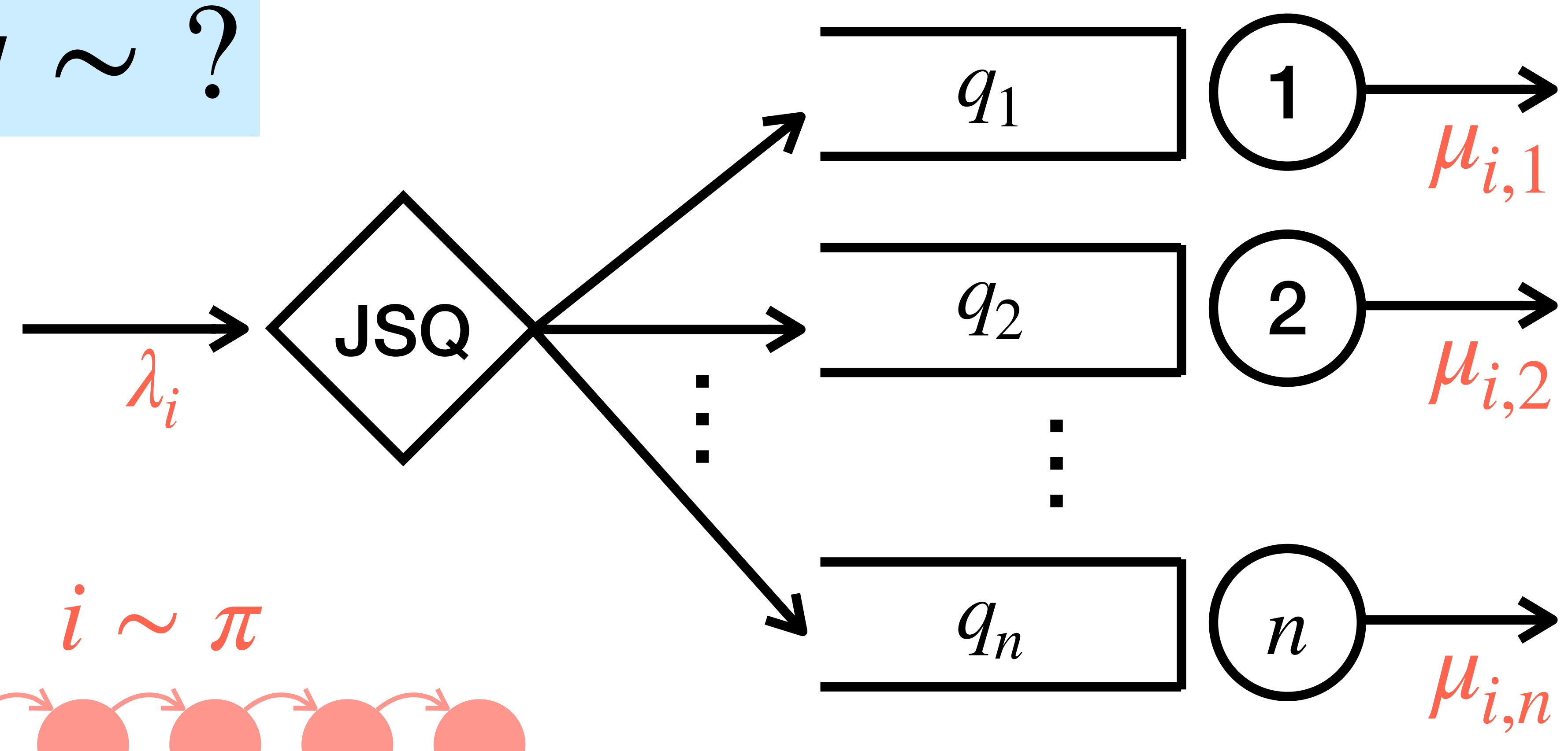

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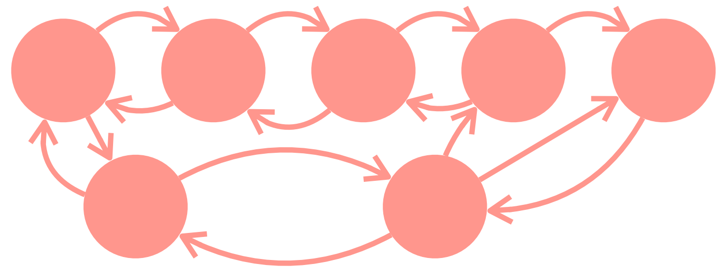

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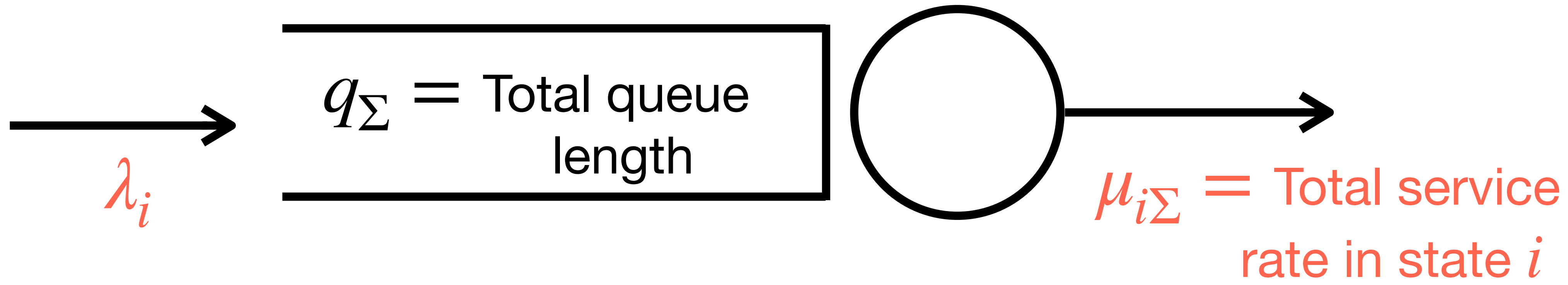

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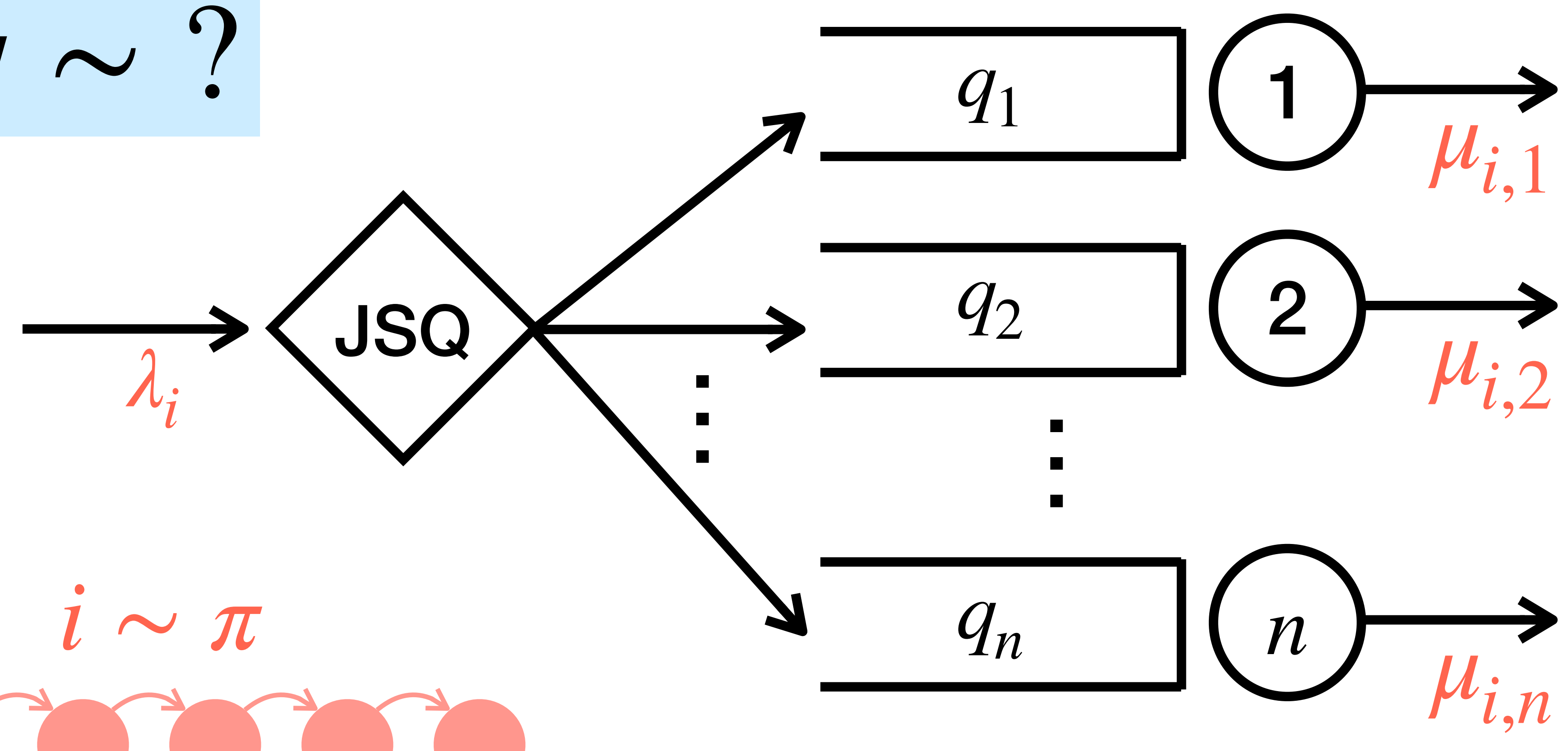

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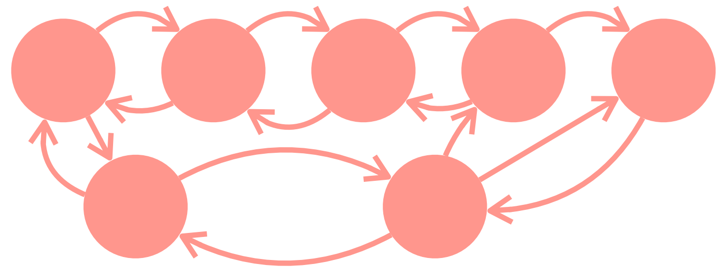

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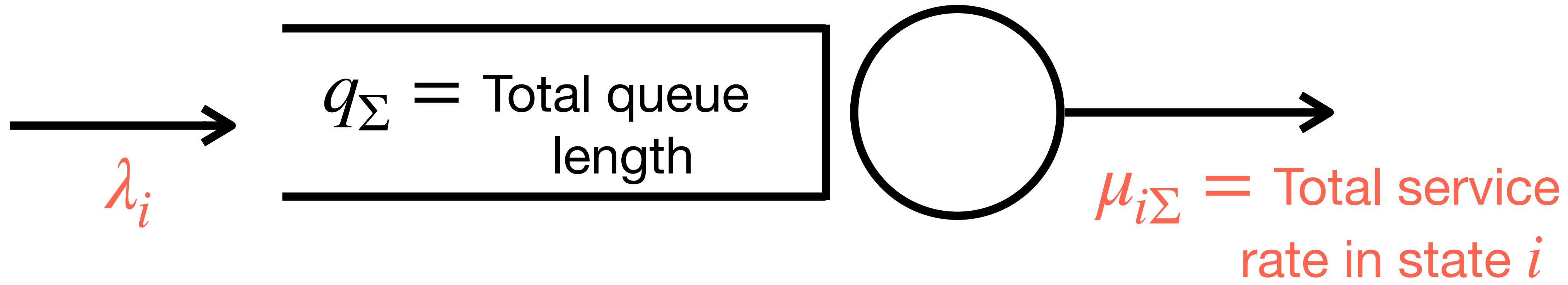

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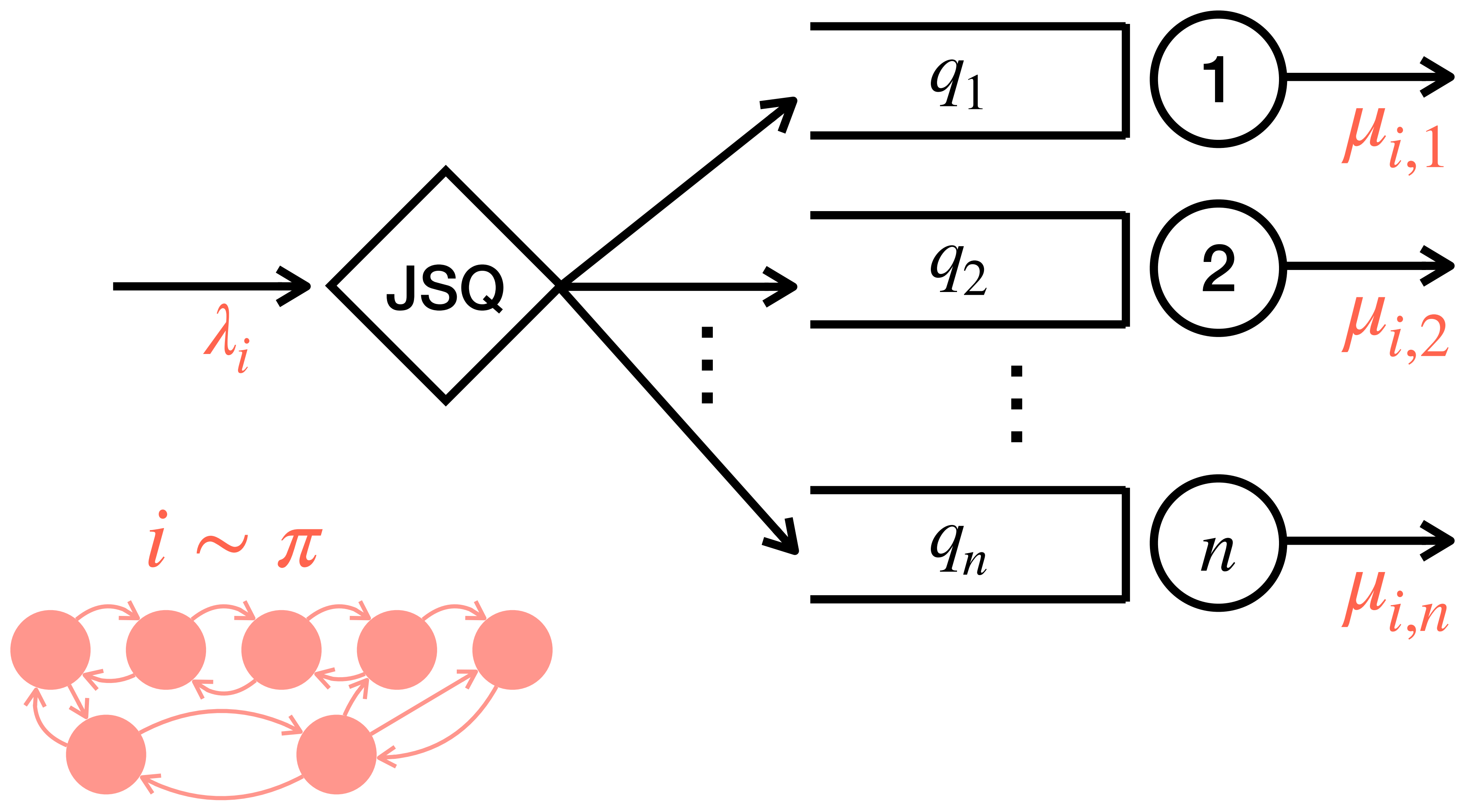

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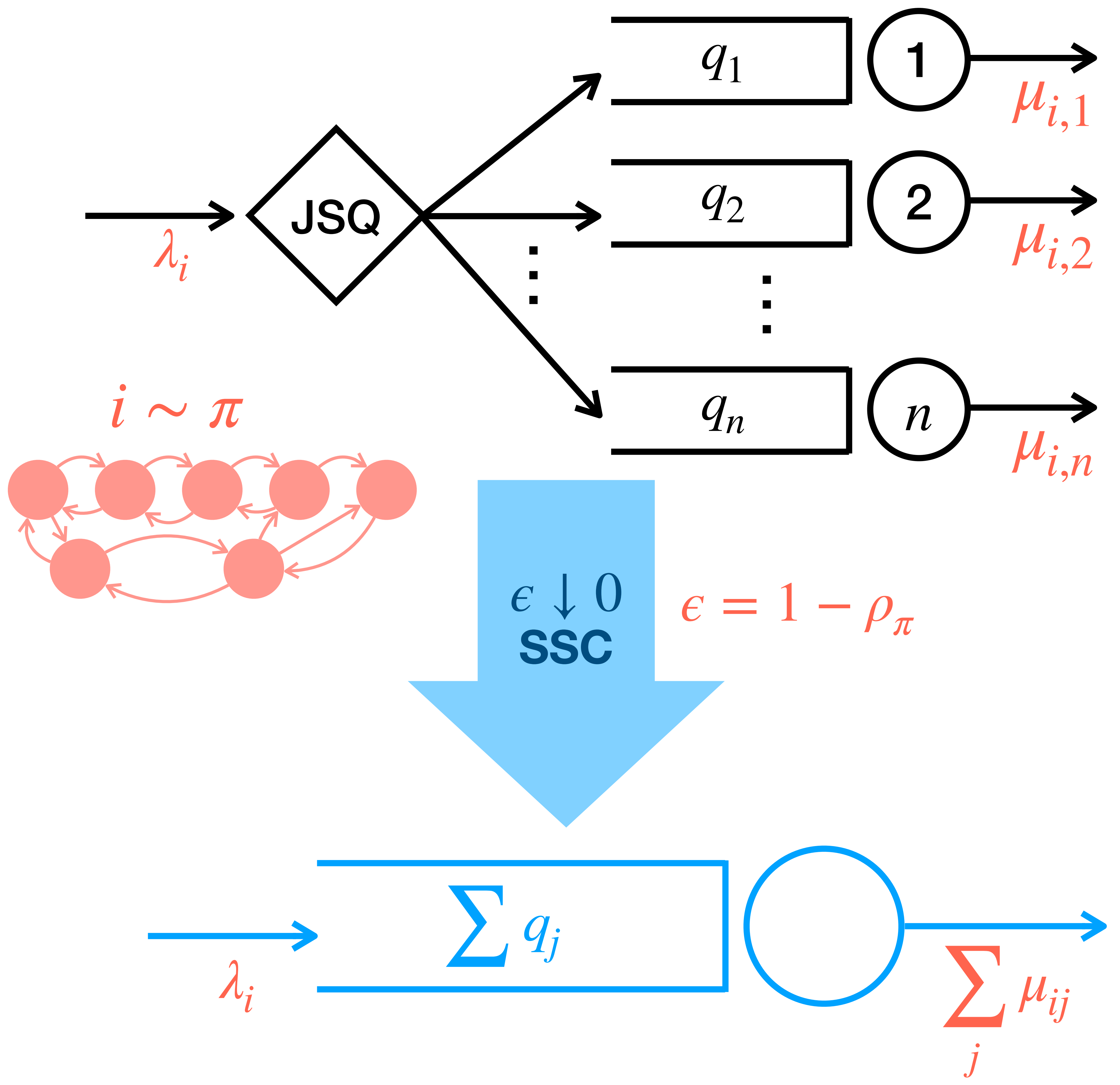

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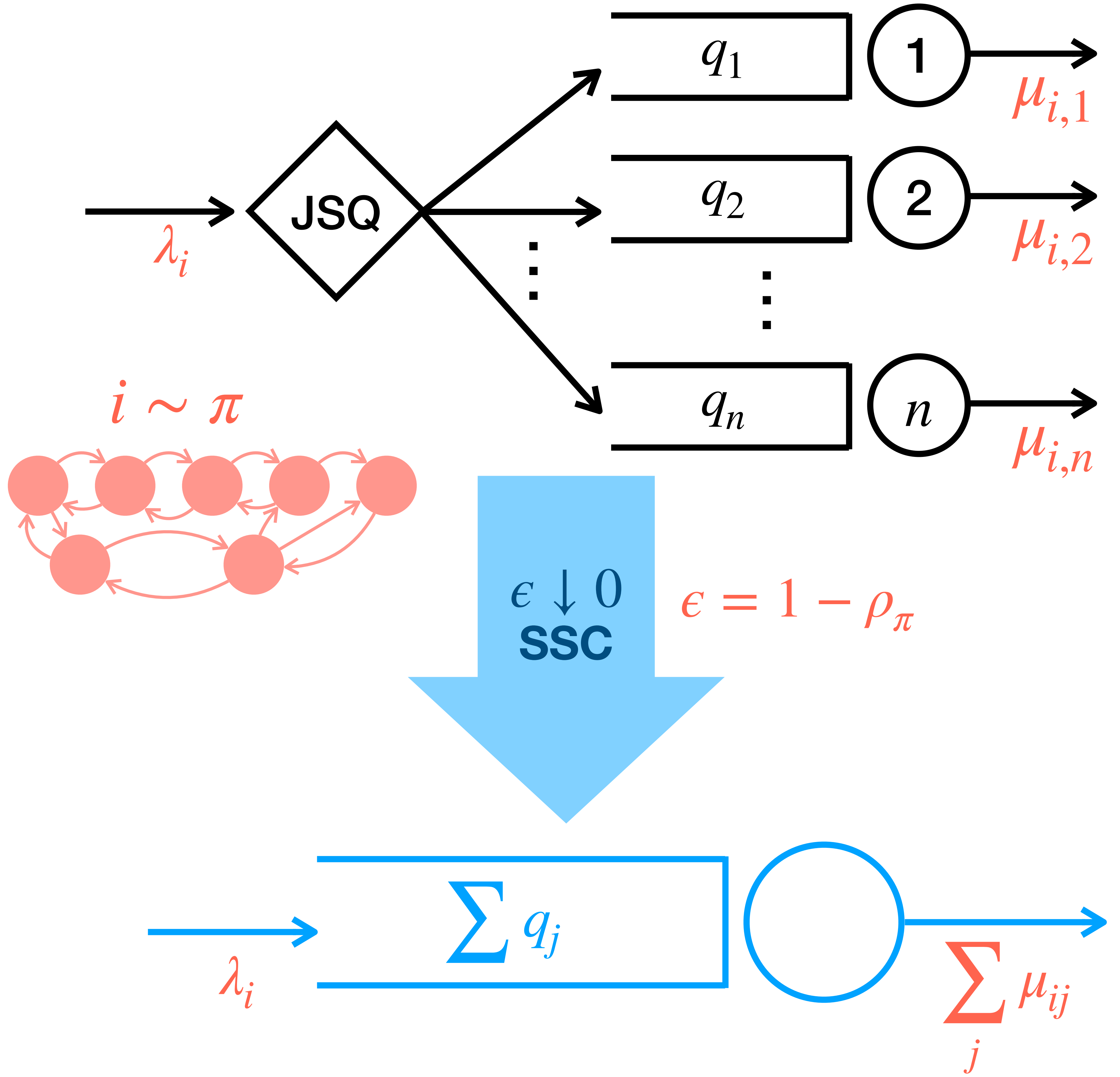

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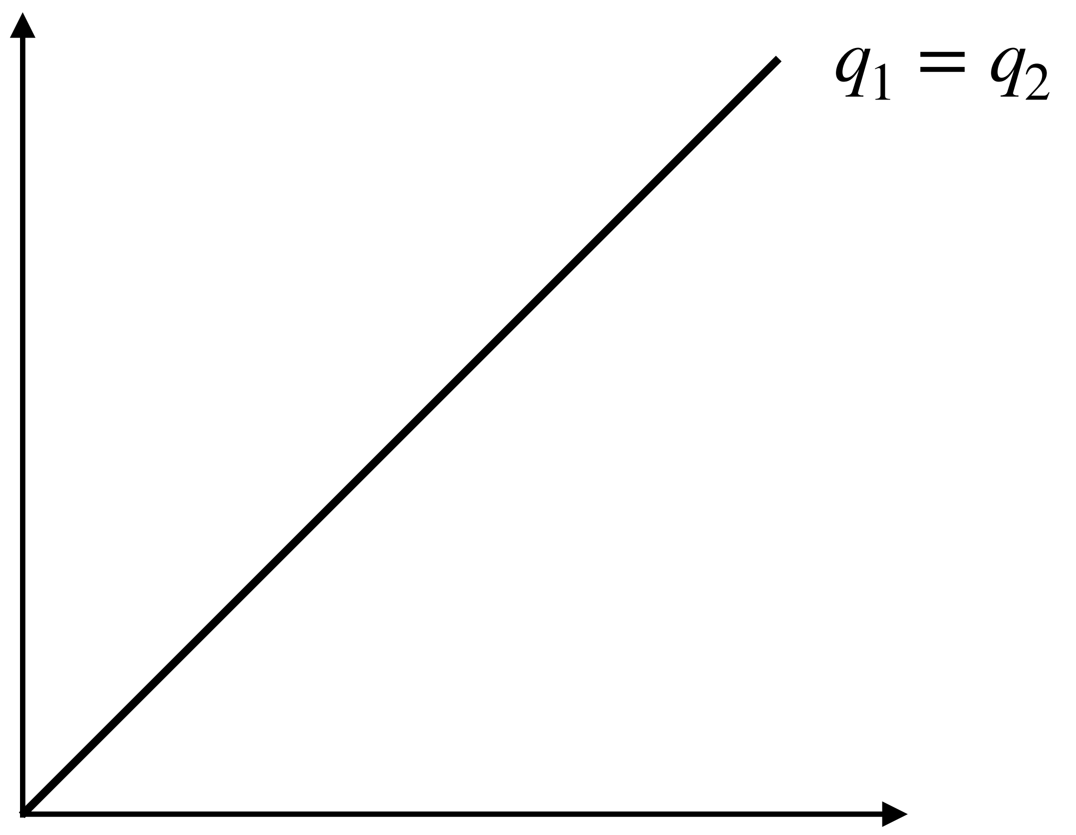

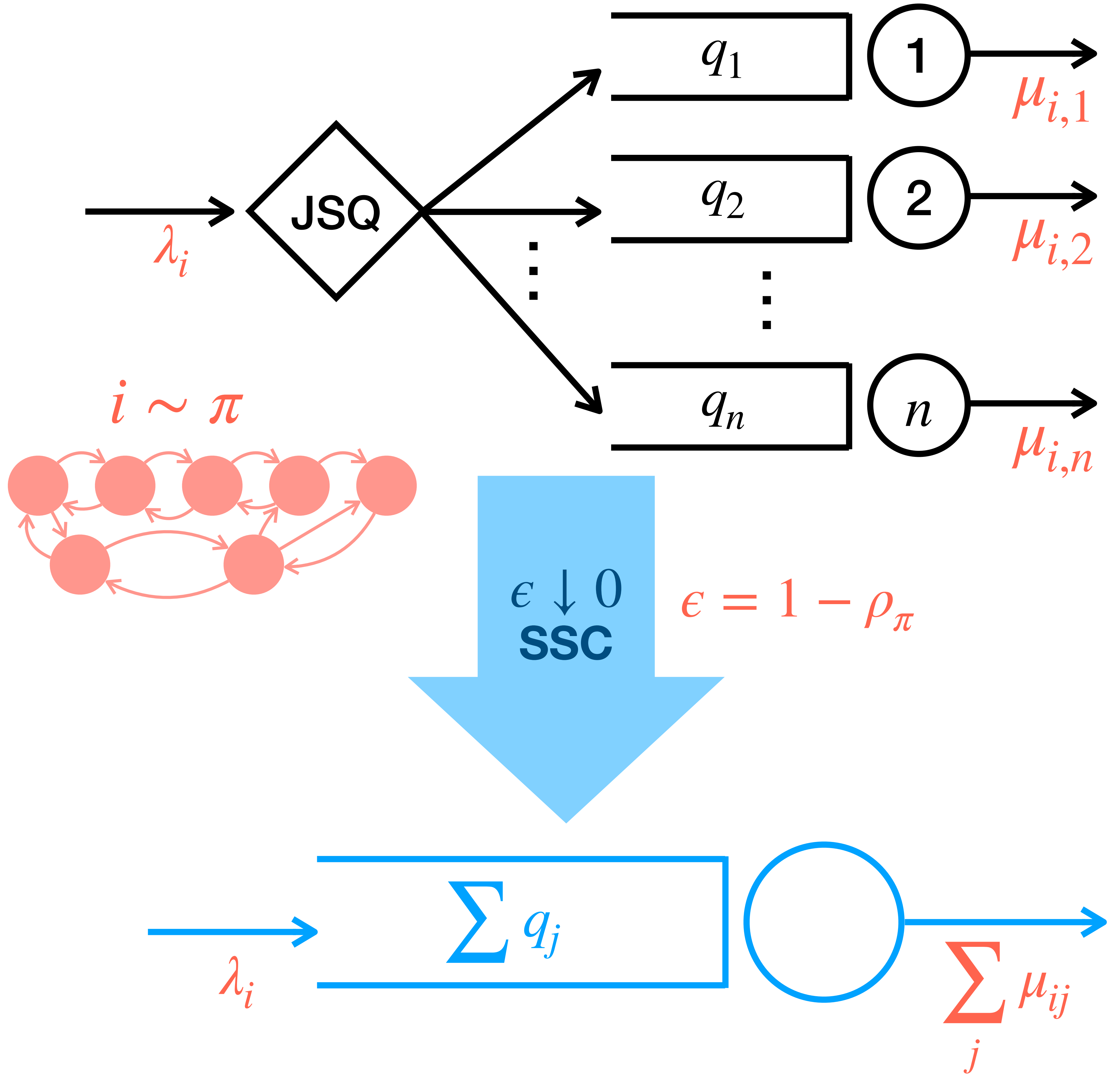

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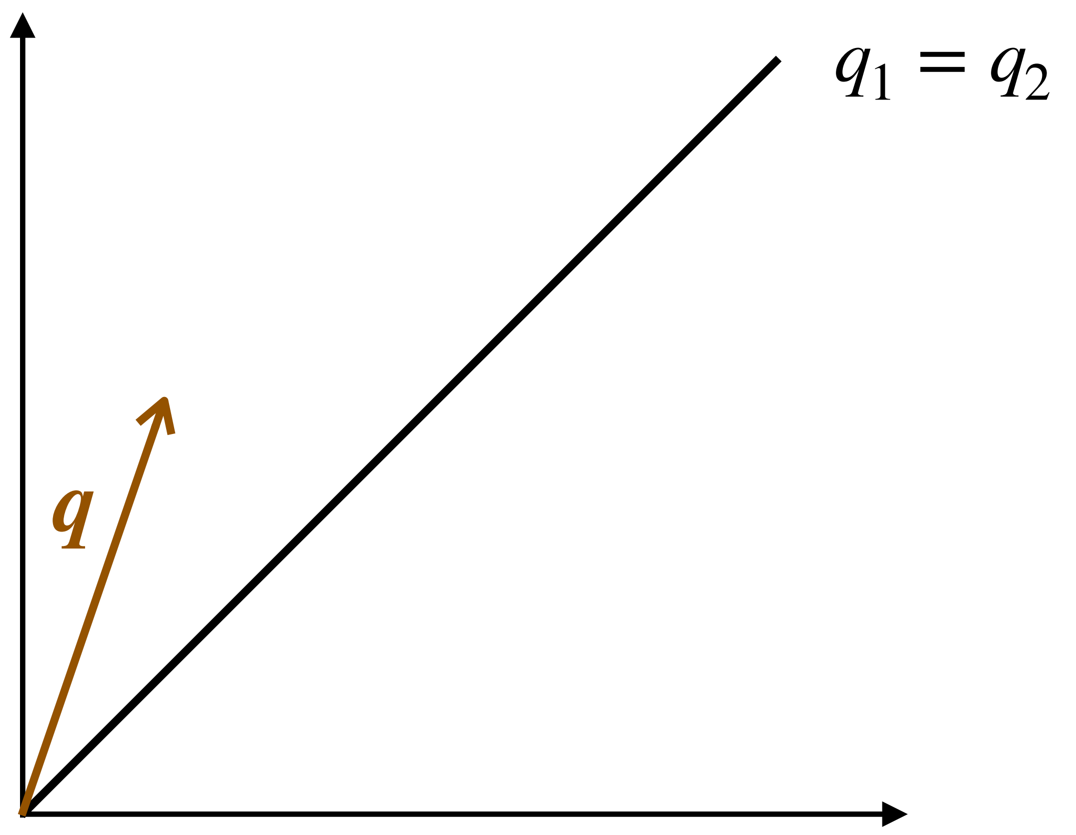

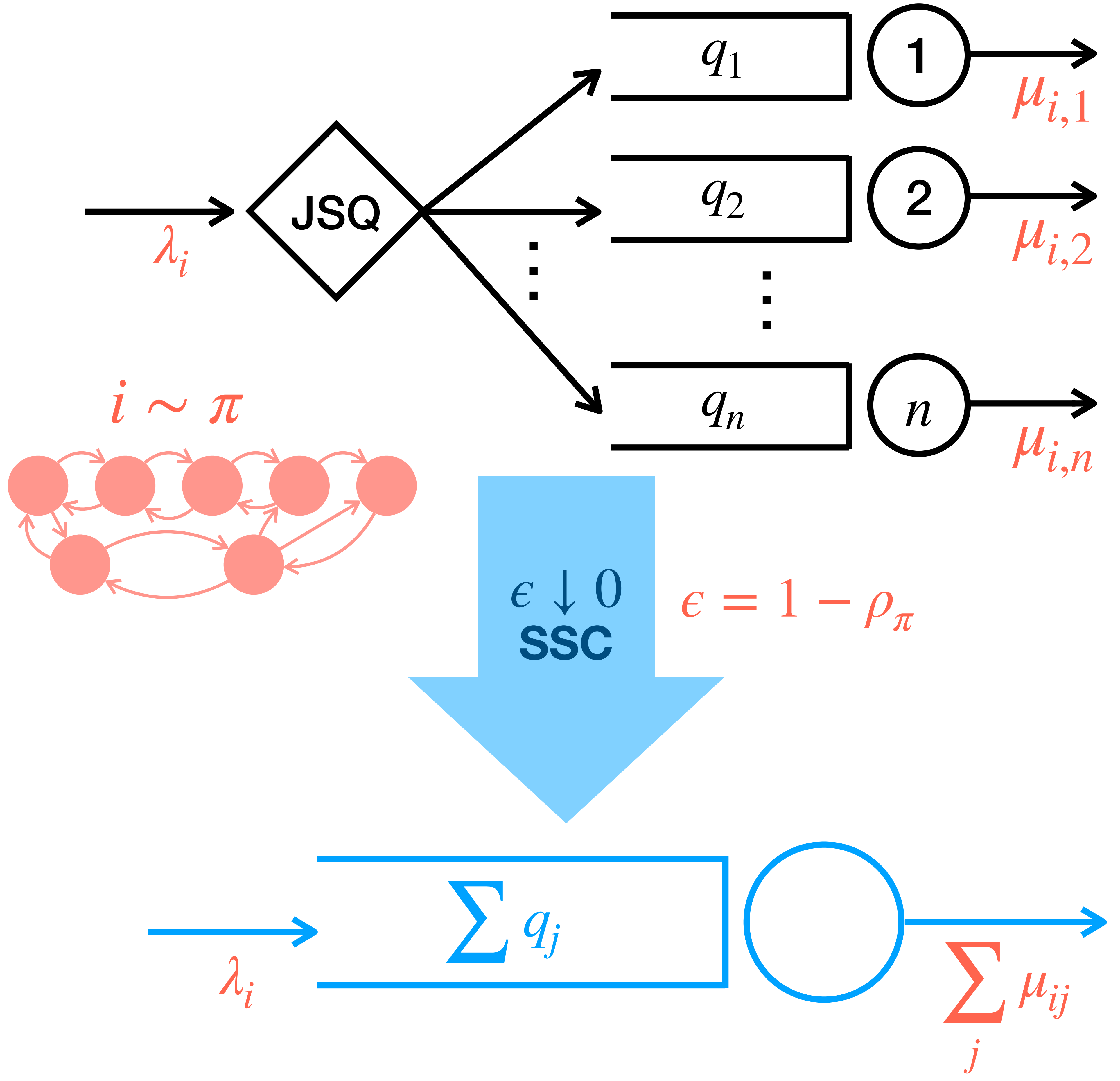

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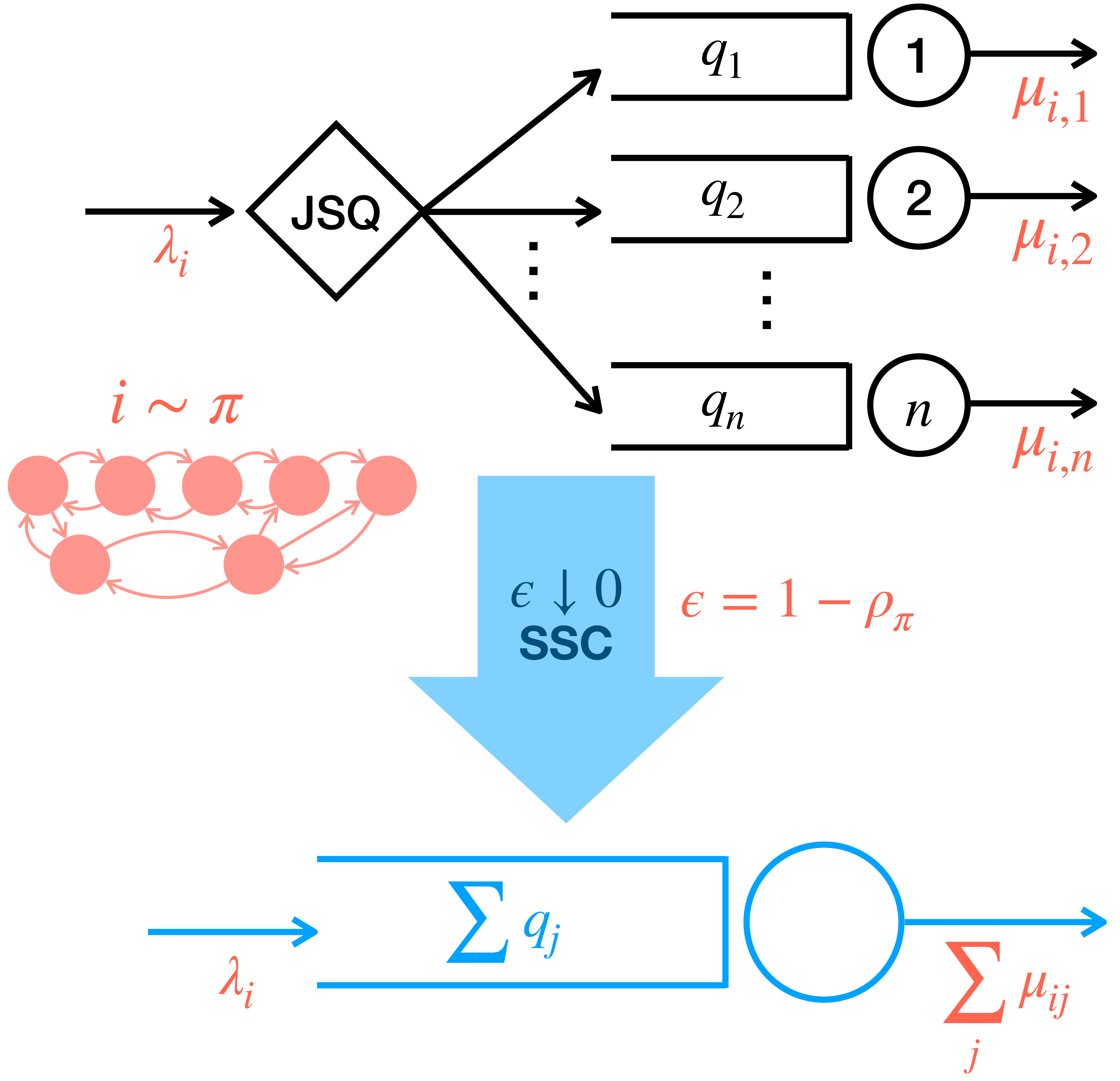

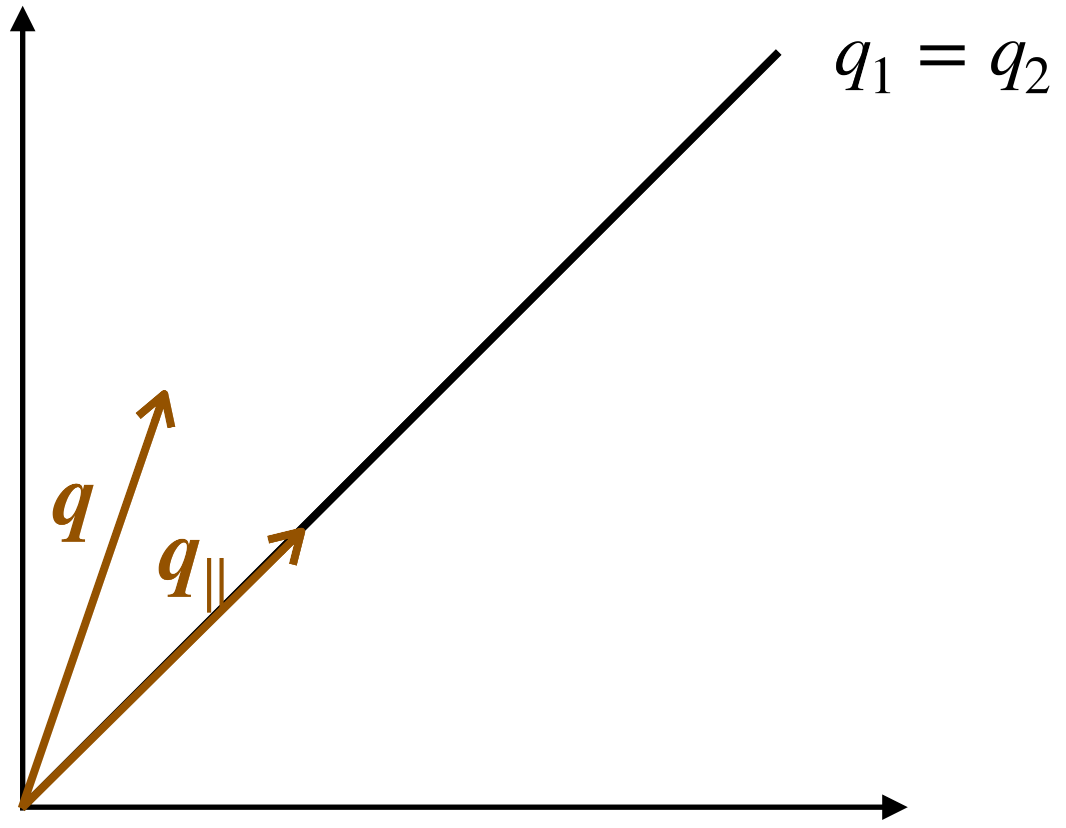

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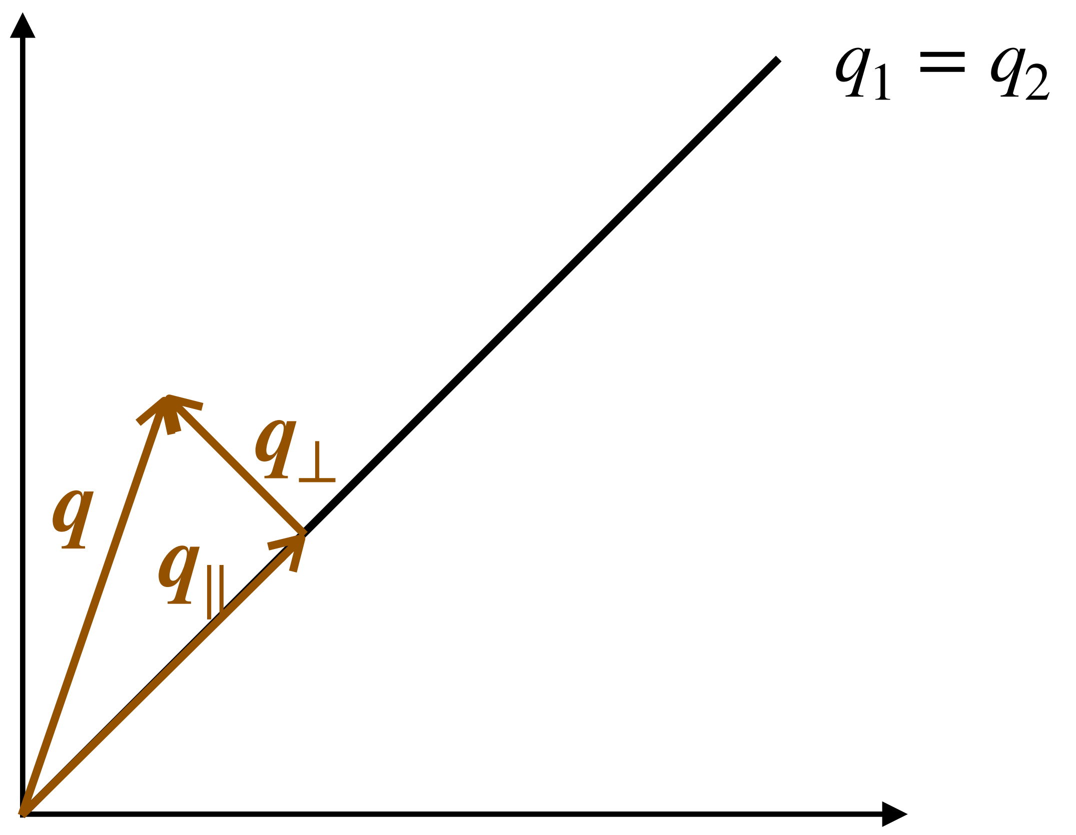

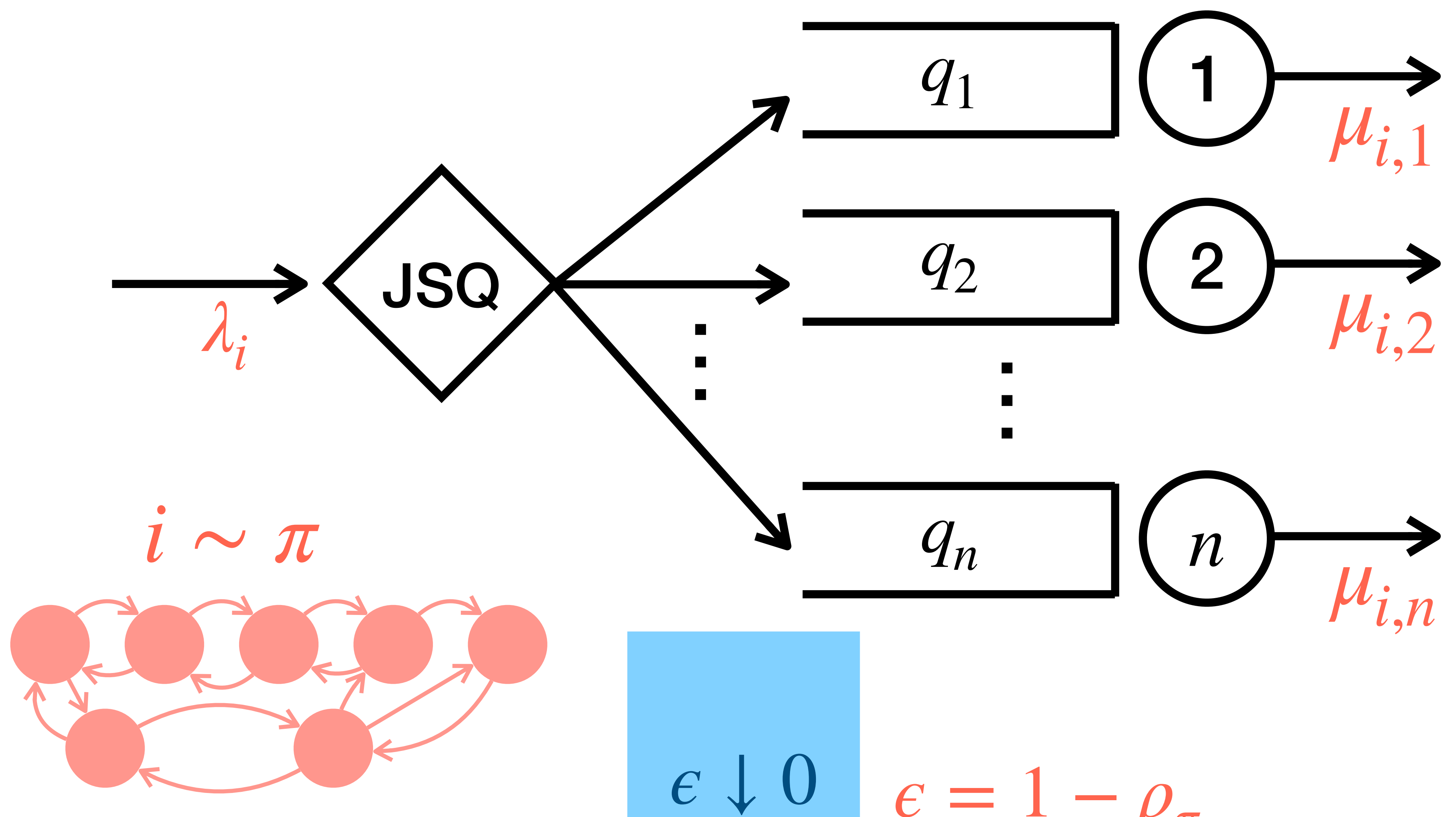

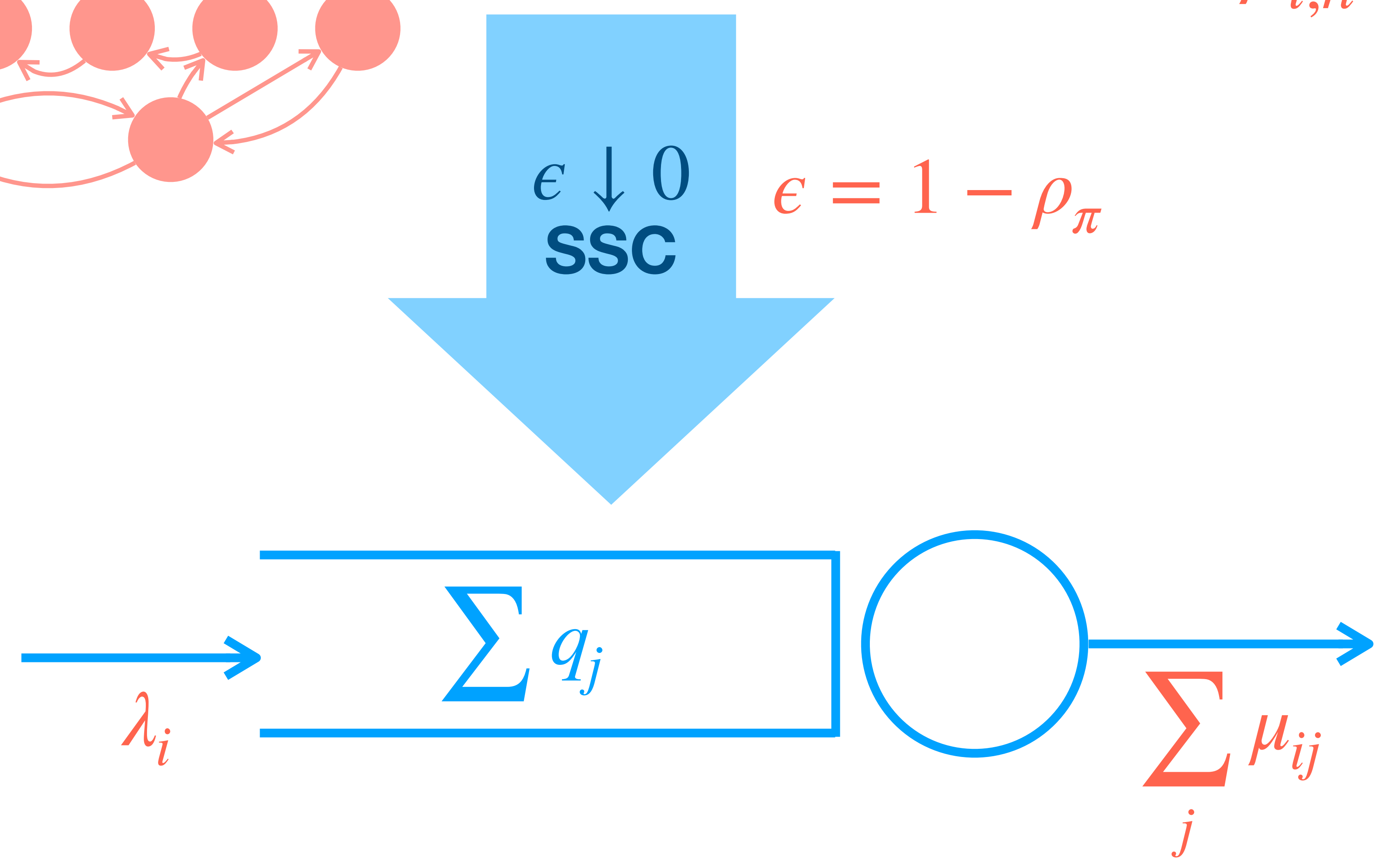

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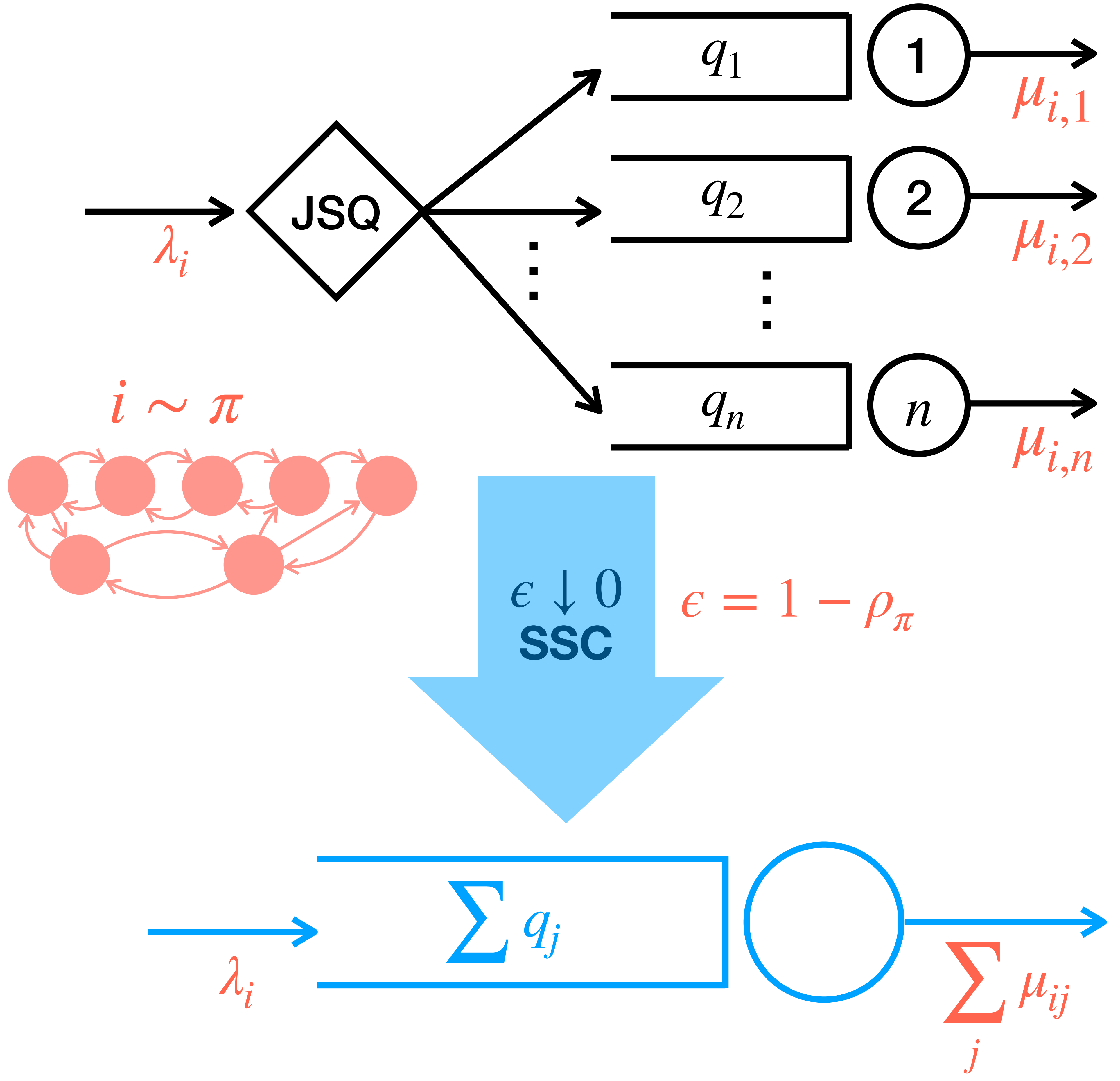

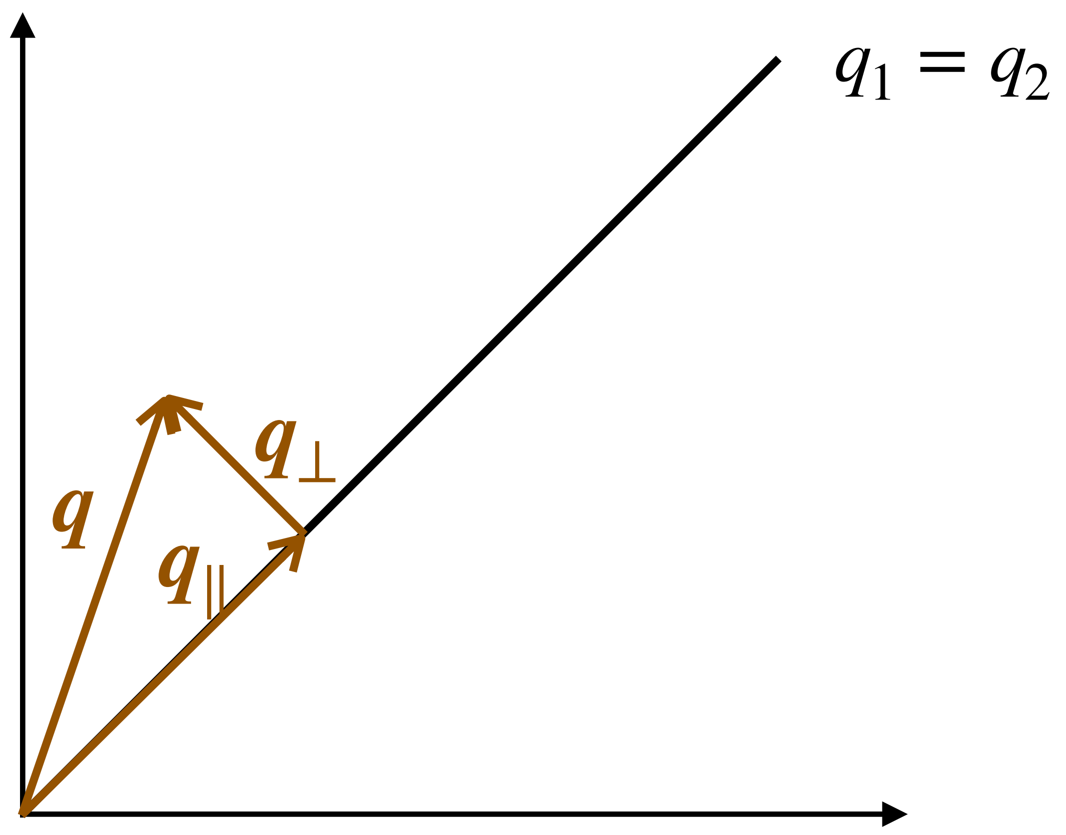

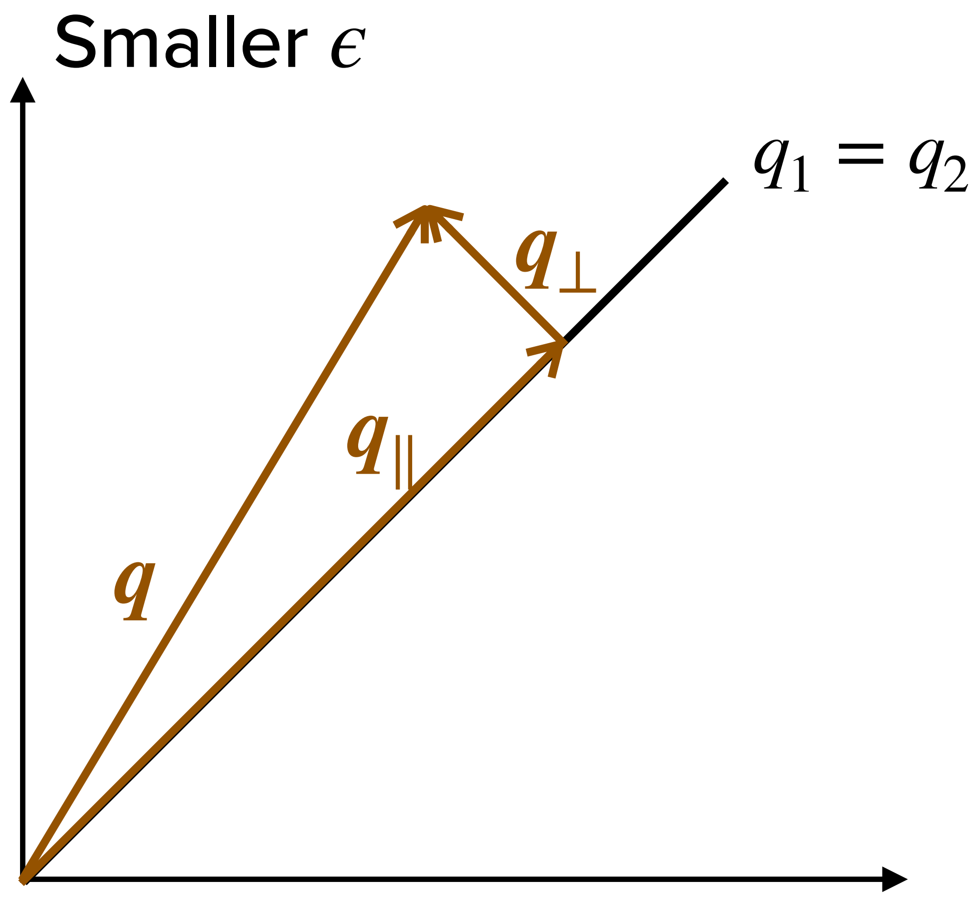

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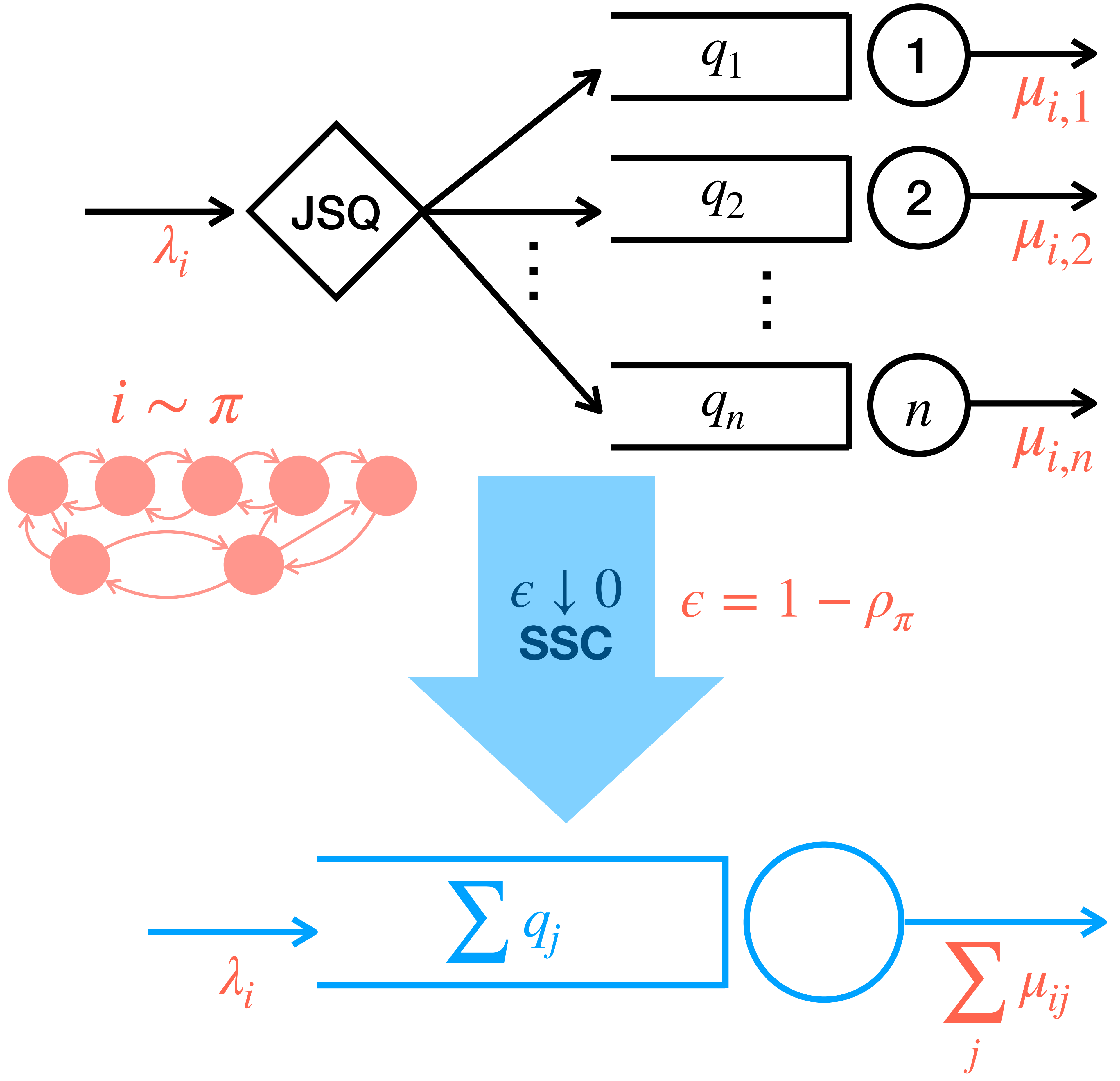

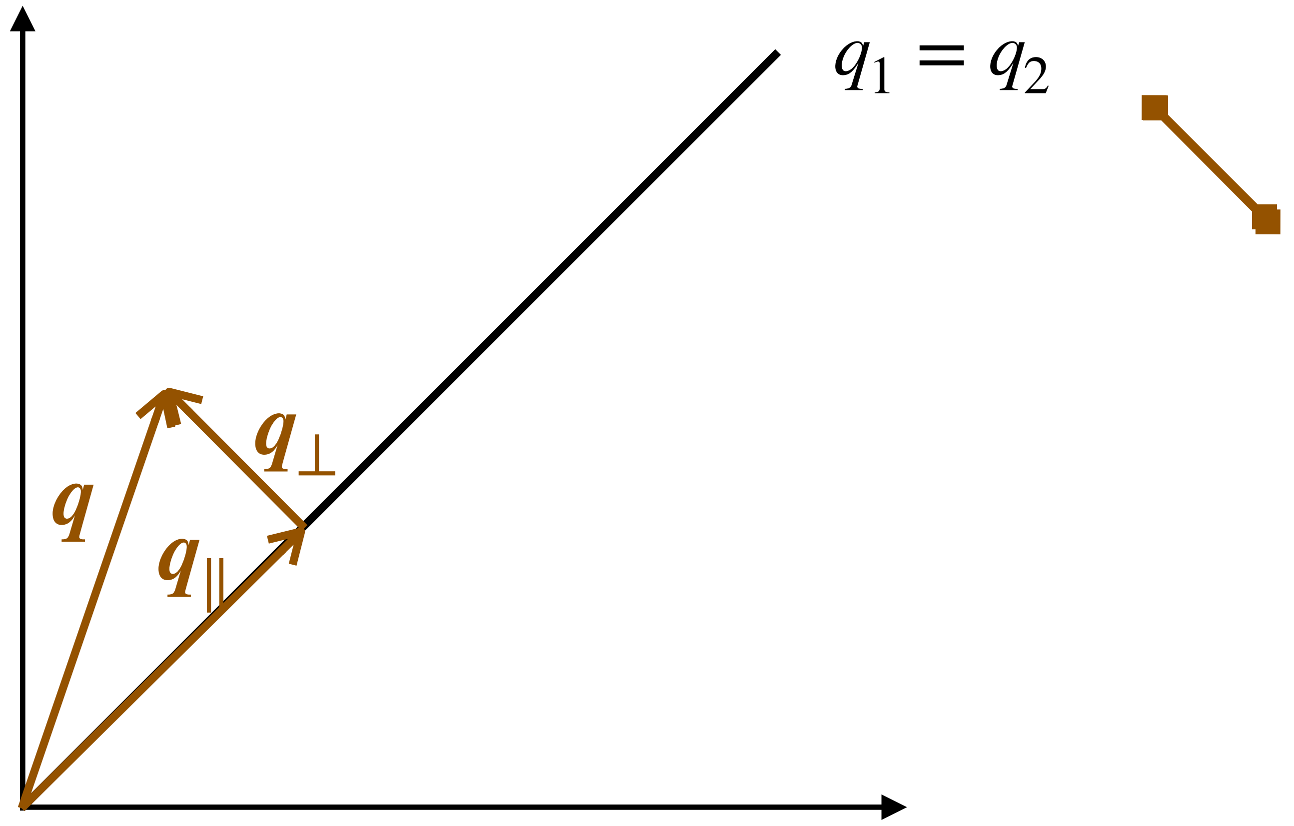

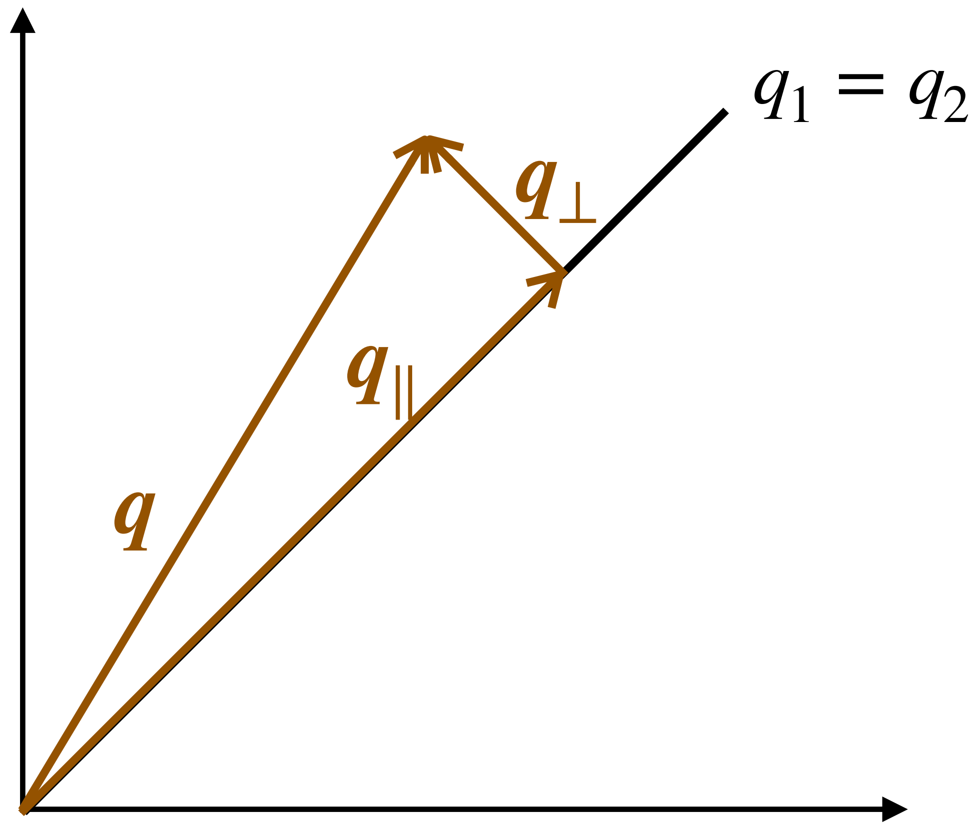

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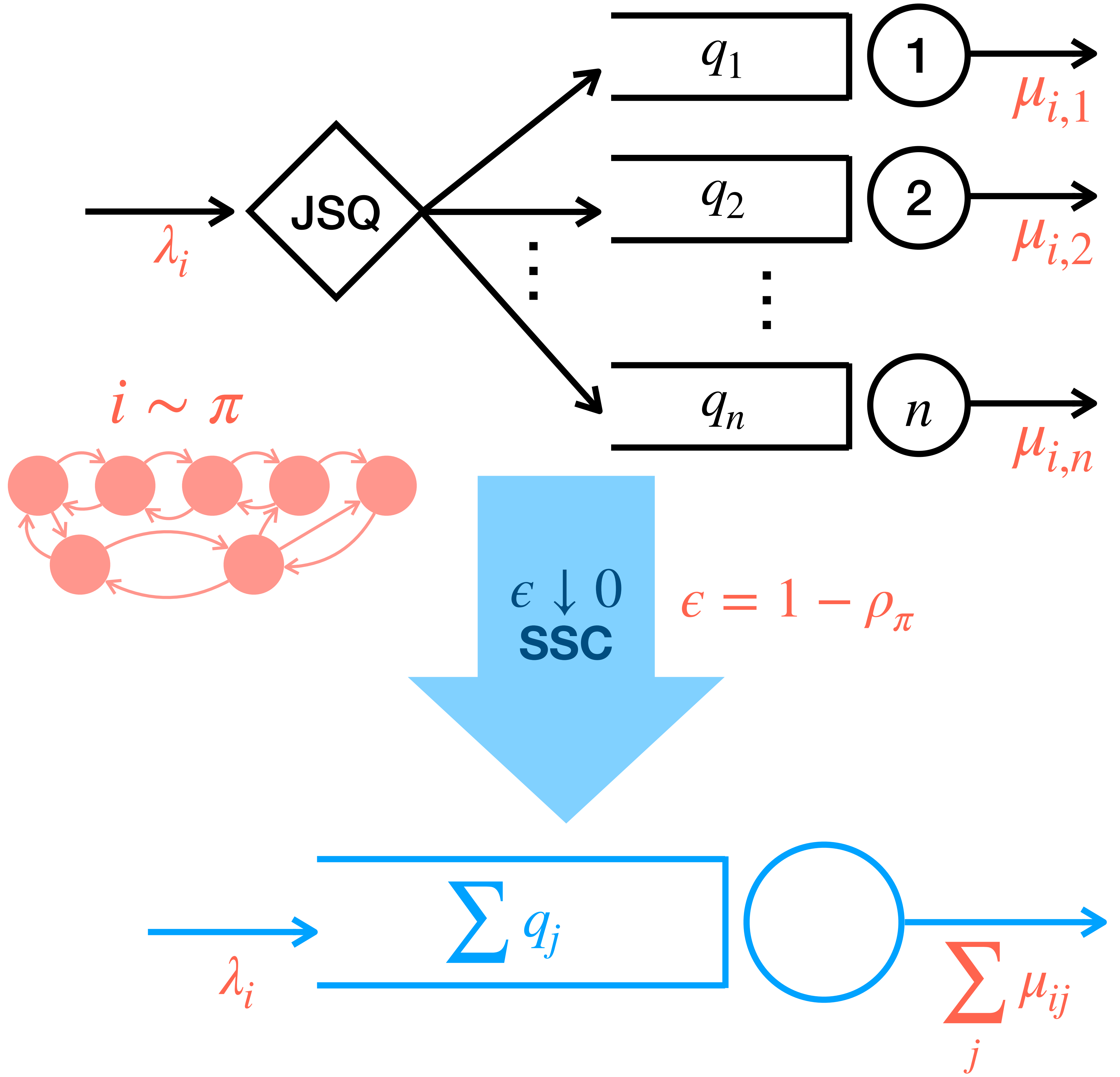

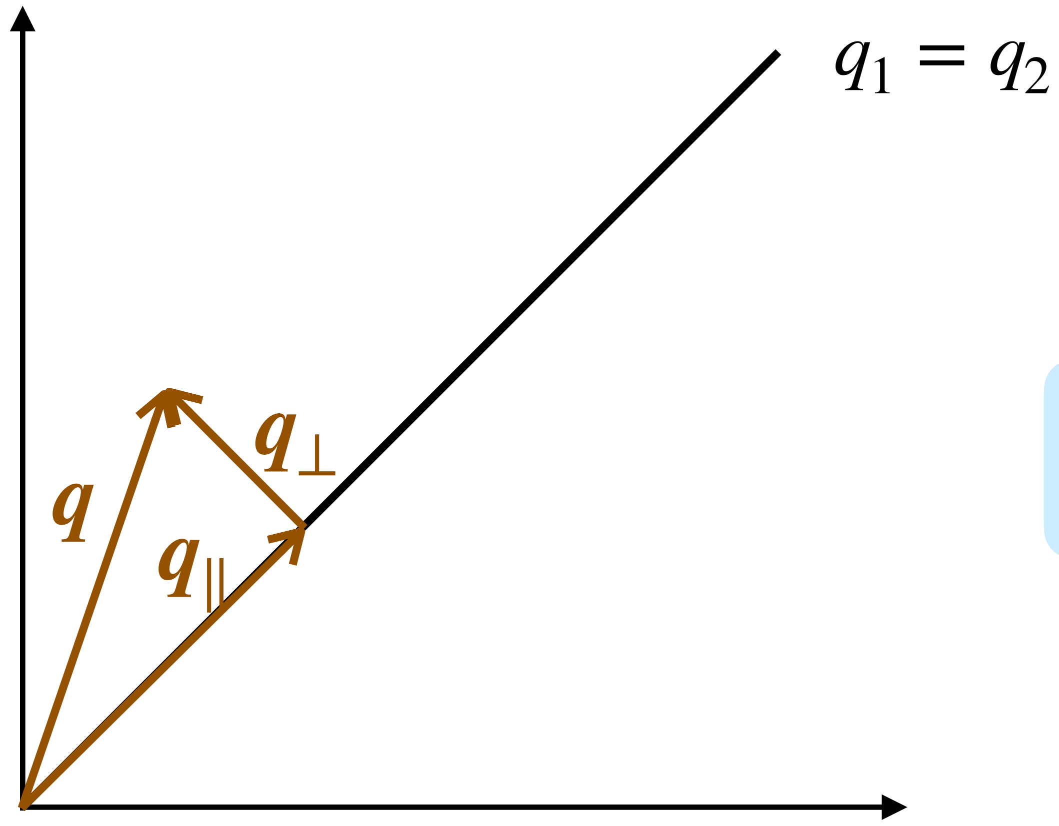

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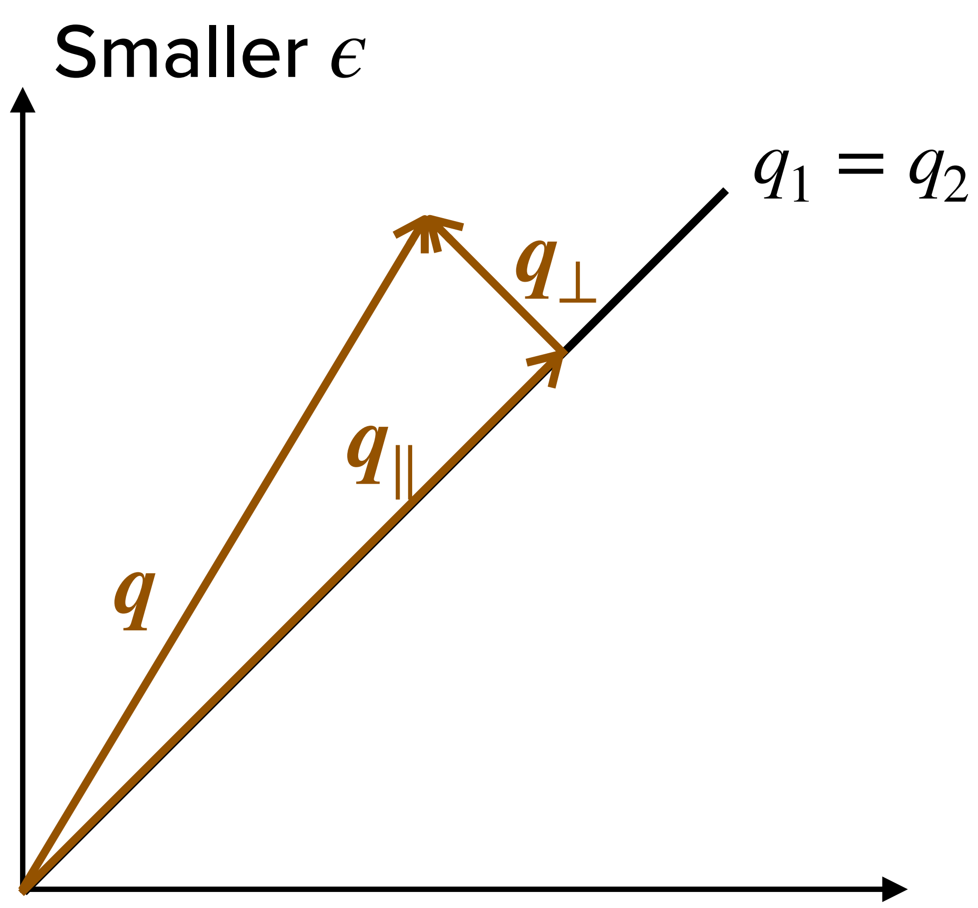

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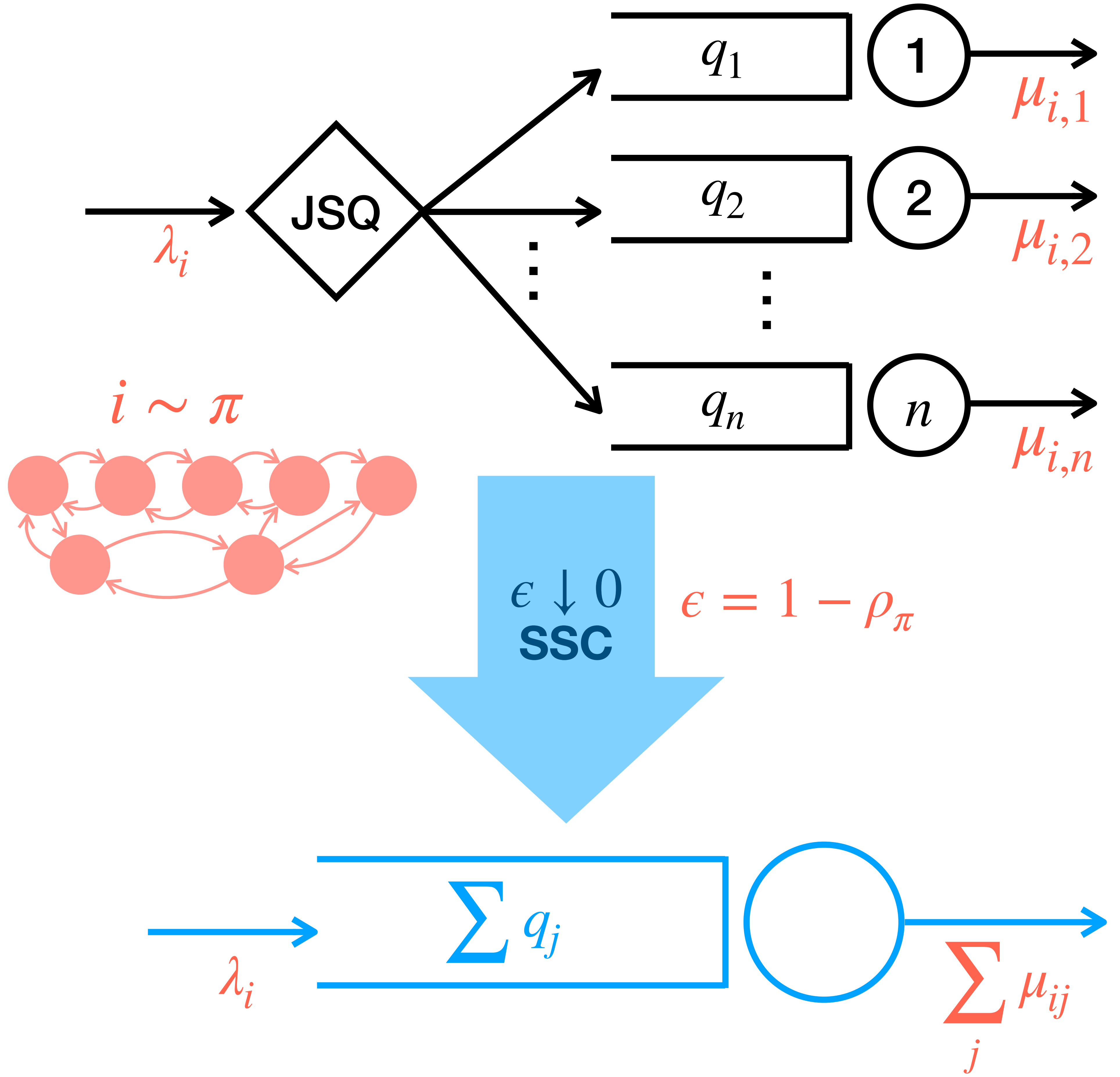
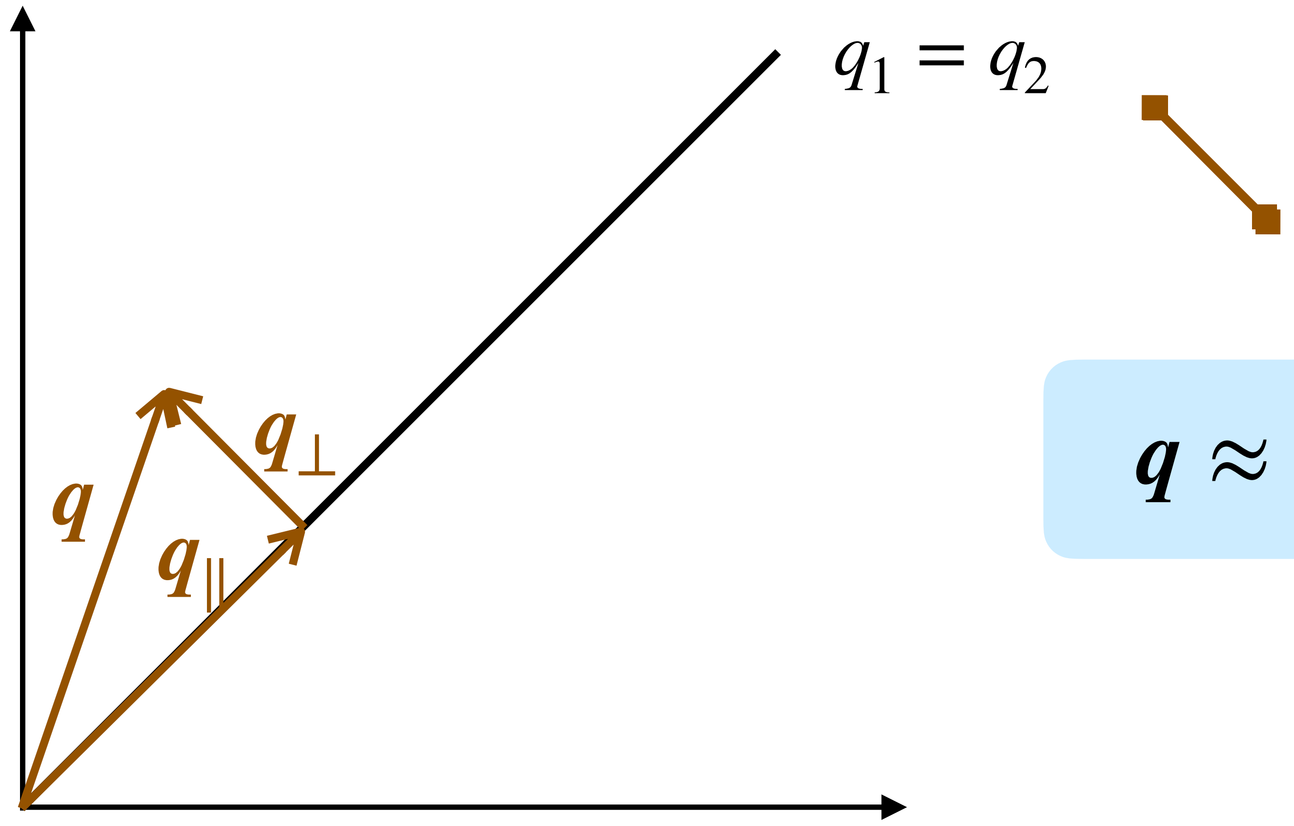
Proof: Drift analysis





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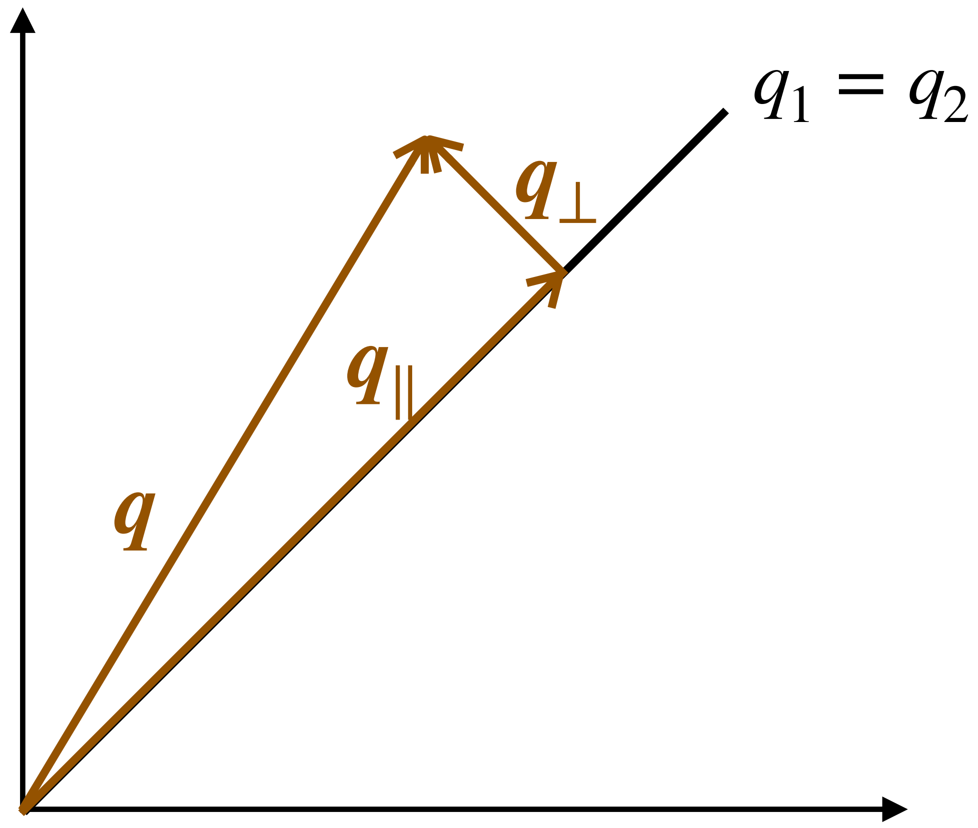
Smaller ϵ



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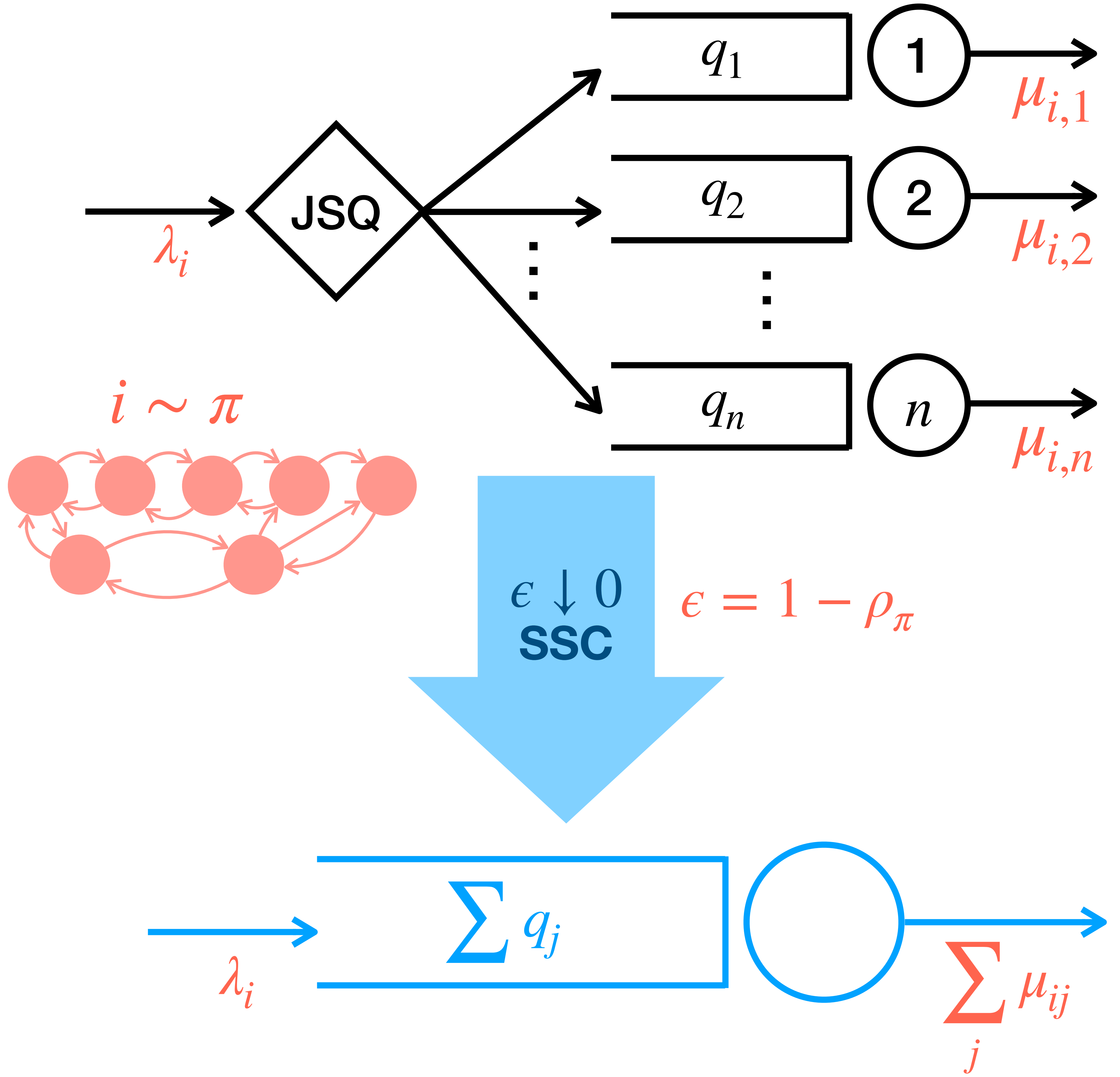

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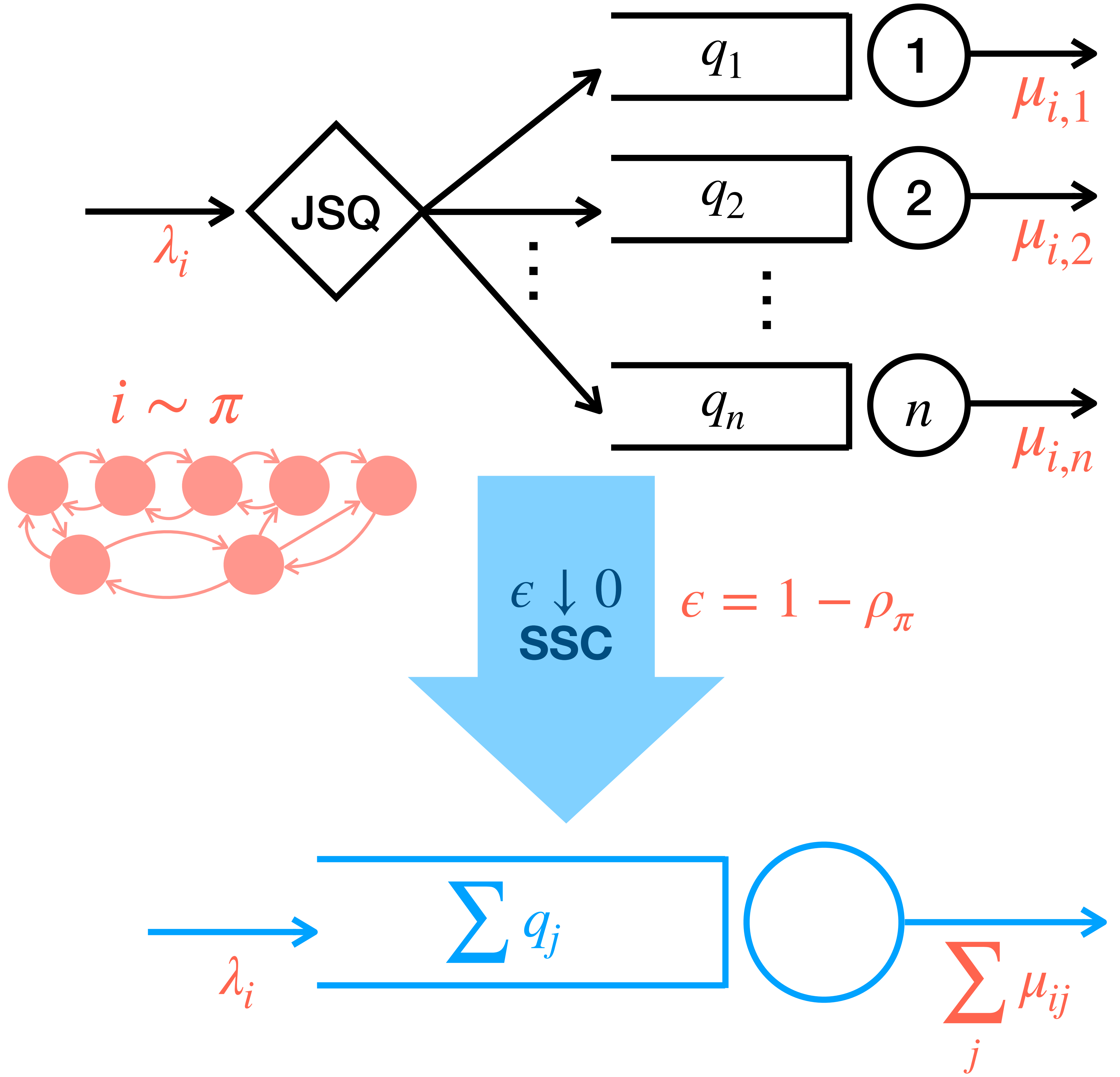

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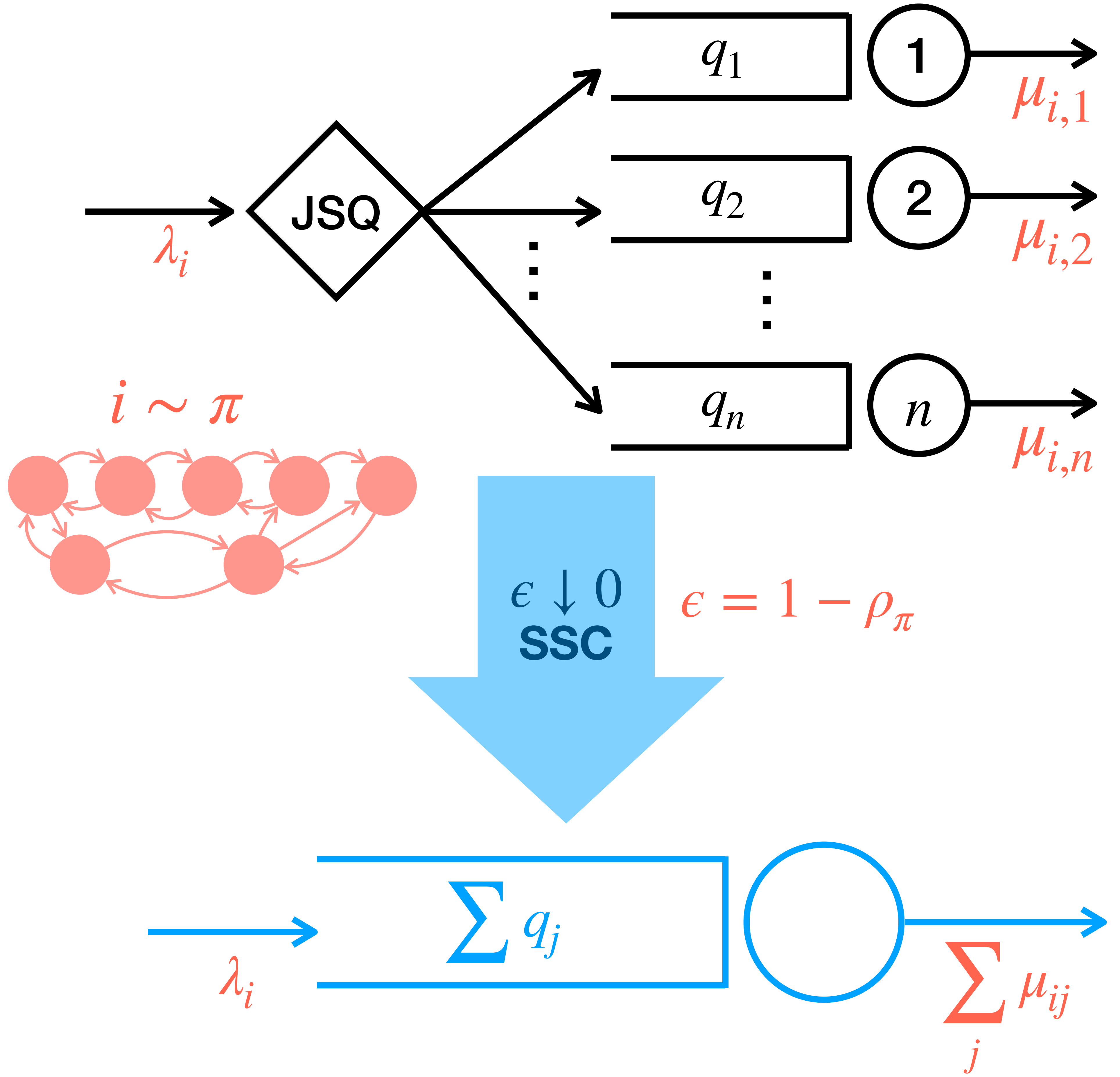

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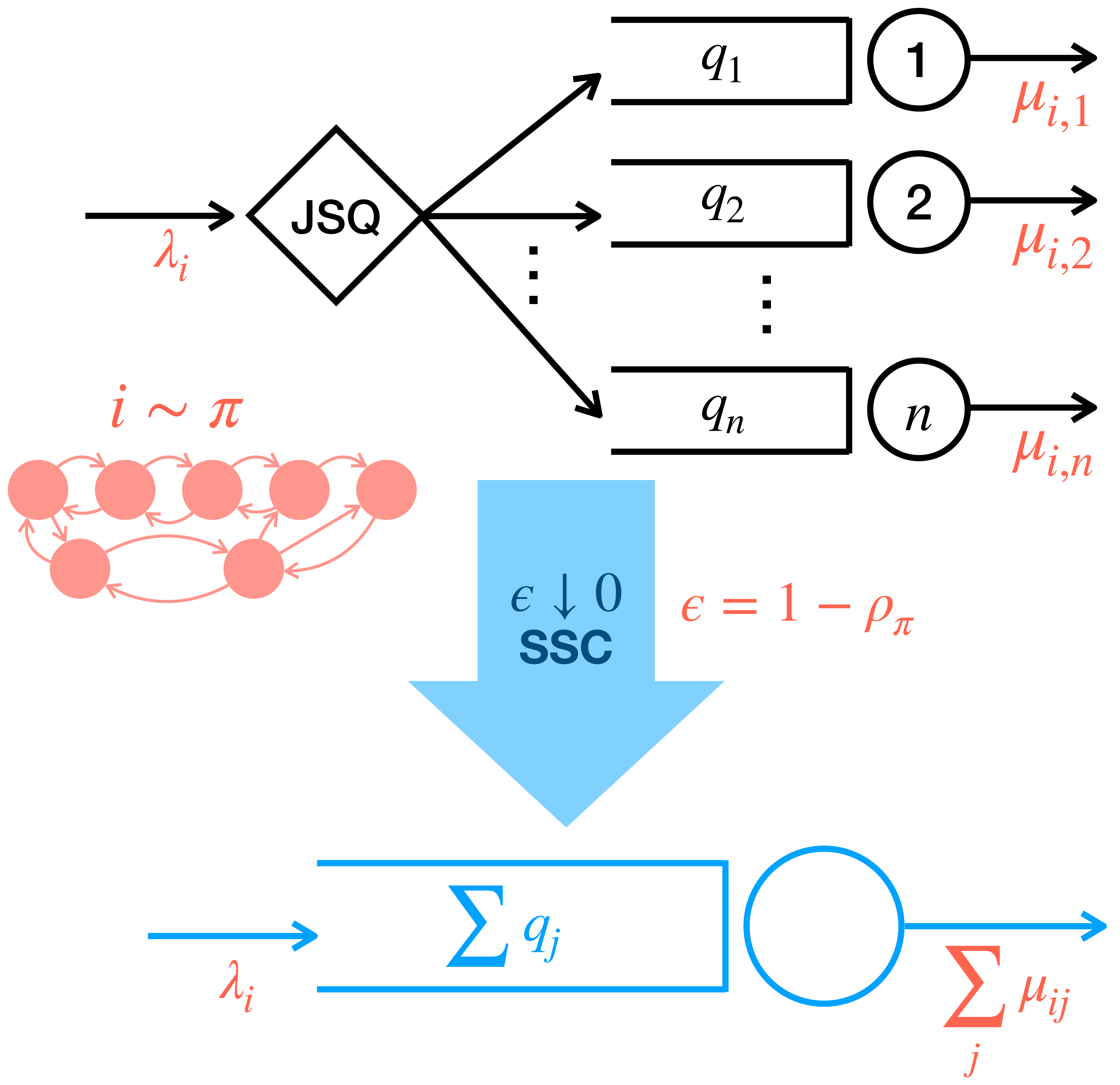

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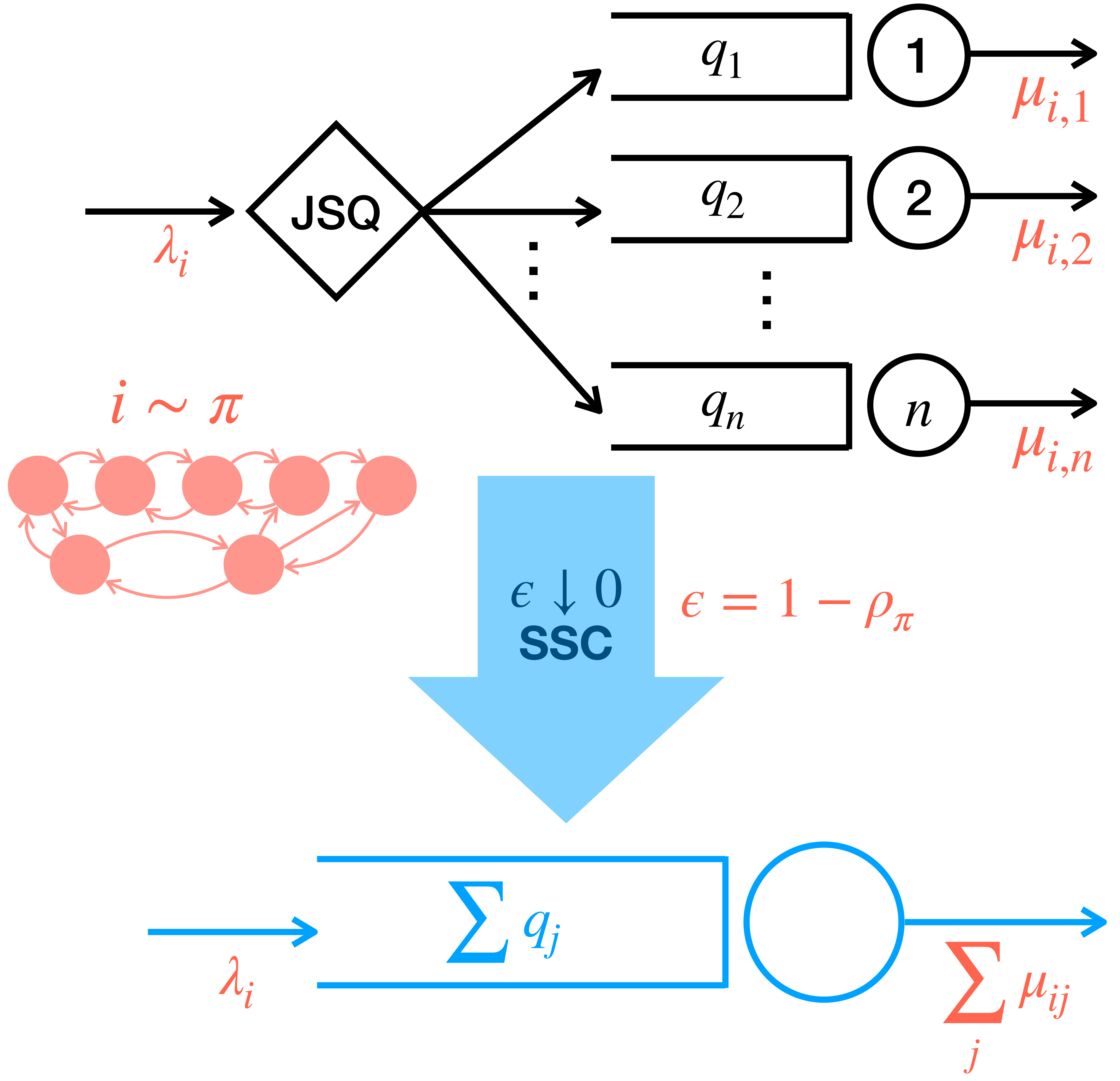

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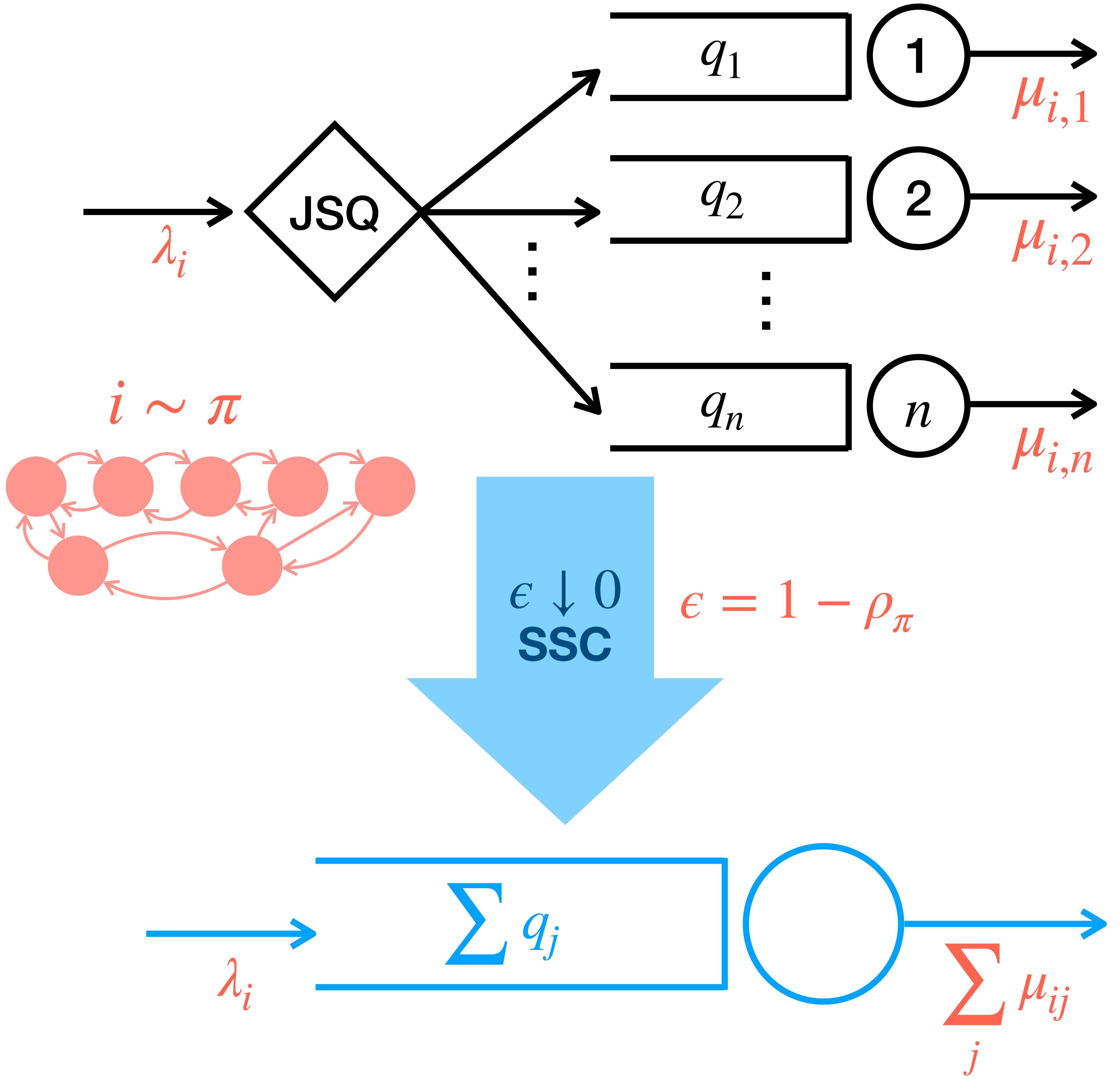

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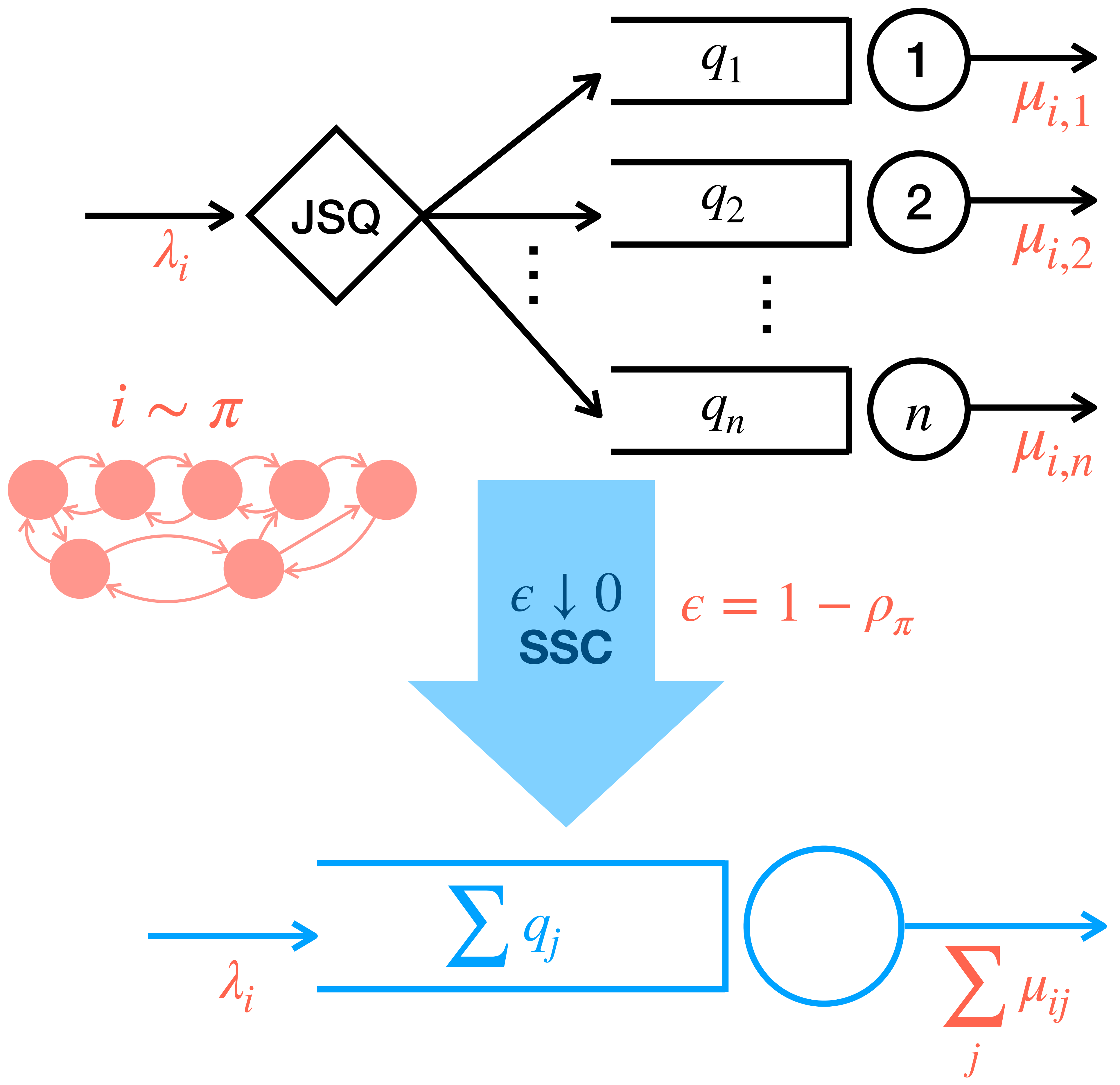

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When does JSQ balance the queue lengths?

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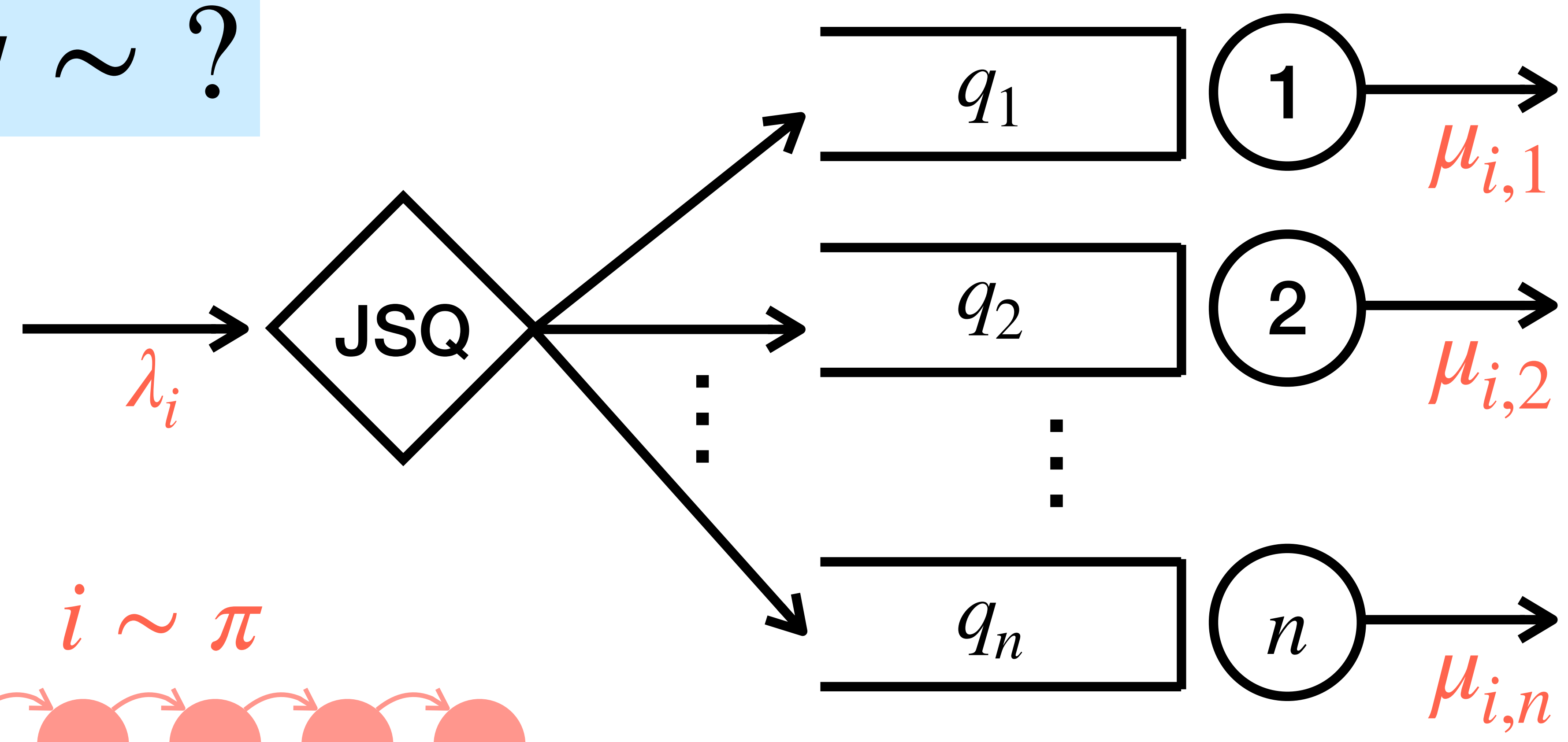

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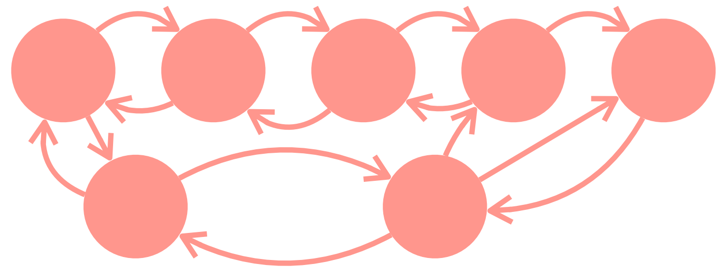

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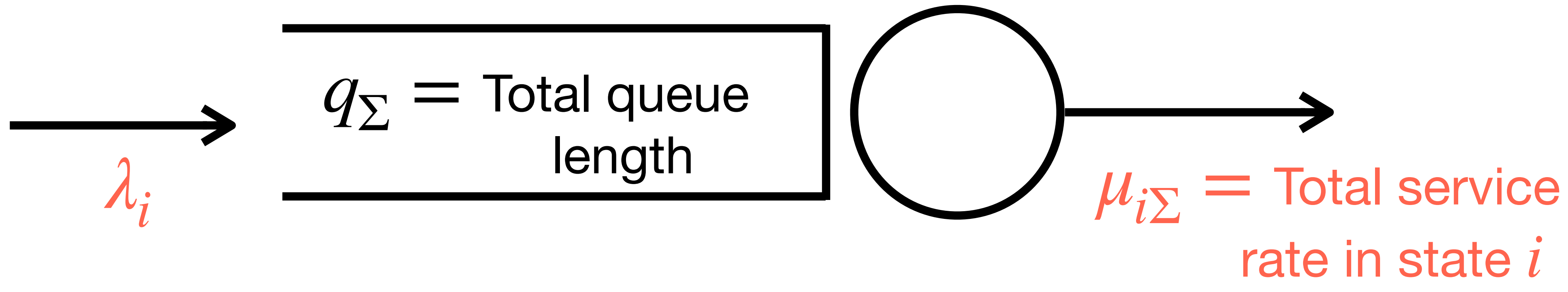

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$$\begin{tikzcd}
\epsilon q^{(\epsilon)}(t) \arrow[r, "t \to \infty"] \arrow[d, "\epsilon \downarrow 0"'] & \epsilon q^{(\epsilon)}(\infty) \\
\tilde{q}(t) &
\end{tikzcd}$$

RBM

Heavy-Traffic Analysis in the Literature

Diffusion Limits Approach

- Most popular
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Interchange of limits

RBM

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Interchange of limits

RBM

Direct Methods

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RBM

Direct Methods

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  Bound Wasserstein distance between $\epsilon q^{(\epsilon)}(\infty)$ and q^*

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  Inductively compute $\mathbb{E}\left[(\epsilon q)^k\right]$ for $k \in \mathbb{N}$

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  Analyze $e^{\epsilon q(\infty)}$ and bound jumps appropriately

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  Analyze drift of $e^{\epsilon q^{(\epsilon)}(\infty)}$ and directly compute $\mathbb{E}\left[e^{\epsilon q^{(\epsilon)}(\infty)}\right]$

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$$

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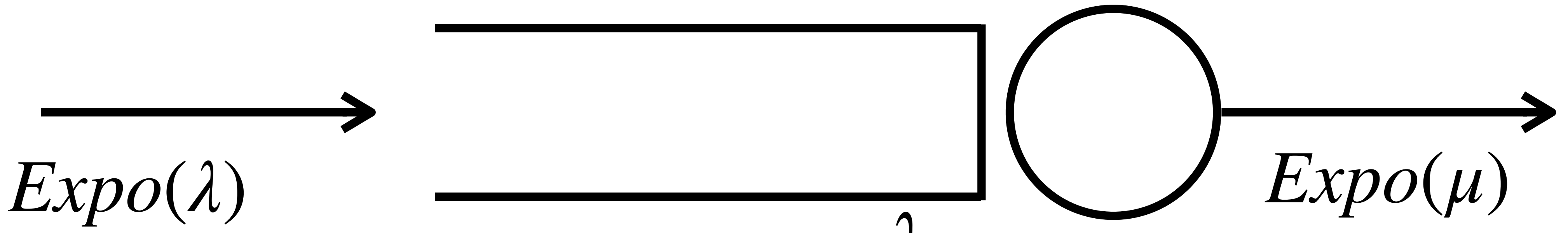
  Analyze drift of $e^{\epsilon q^{(\epsilon)}(\infty)}$ and directly compute $\mathbb{E}\left[e^{\epsilon q^{(\epsilon)}(\infty)}\right]$
  - ✓ Tractable analysis
  - ✓ Compute distribution
  - ✓ Obtain tail bounds

Transform Method for $M/M/1$ queue

Transform Method for $M/M/1$ queue

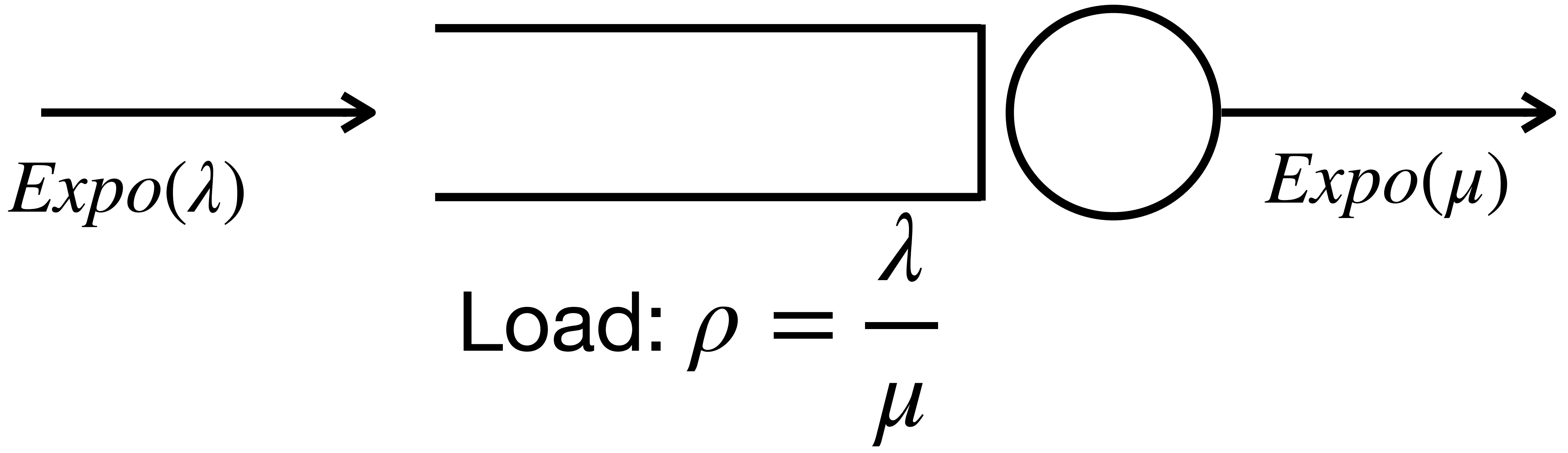


Load: $\rho = \dfrac{\lambda}{\mu}$

Transform Method for $M/M/1$ queue

Step 1: Drift of exponential test function

$\varphi_\theta(q) = e^{\theta q}$, $\theta \in \mathbb{R}$



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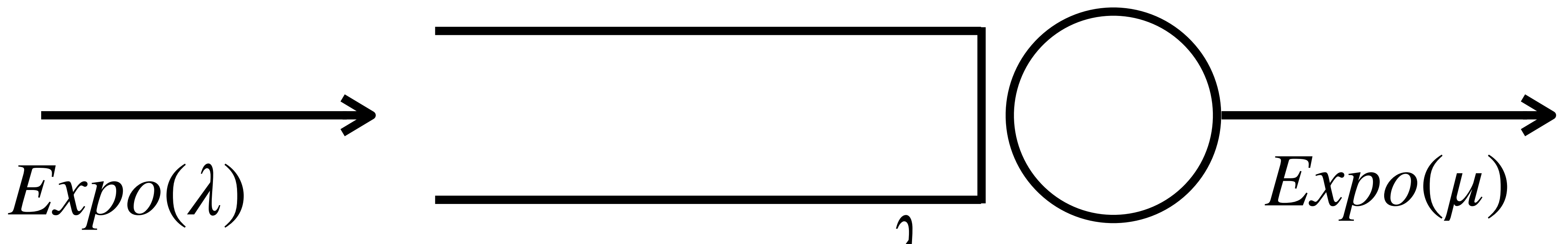

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$\Delta\varphi_\theta(q) = \mathbb{E}\left[e^{\theta q(t+)} - e^{\theta q} \,\middle|\, q(t) = q\right]$

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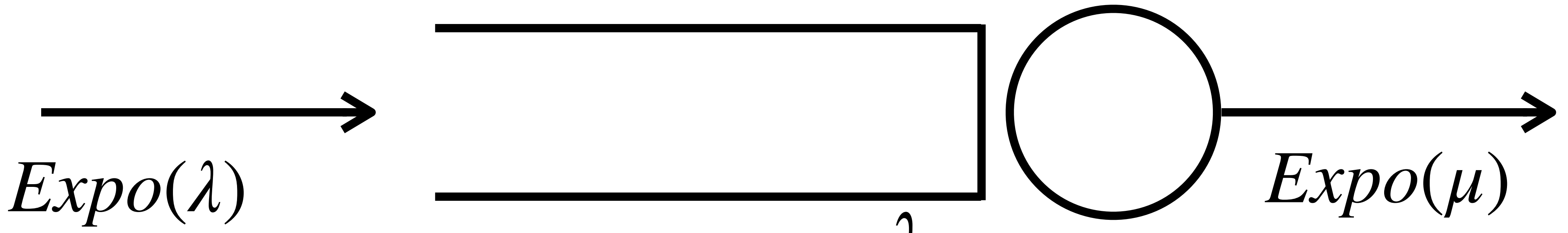
Arrival / Departure

$$\Delta\varphi_\theta(q) = \lambda(e^{\theta(q+1)} - e^{\theta q}) + \mu 1_{\{q>0\}}(e^{\theta(q-1)} - e^{\theta q})$$



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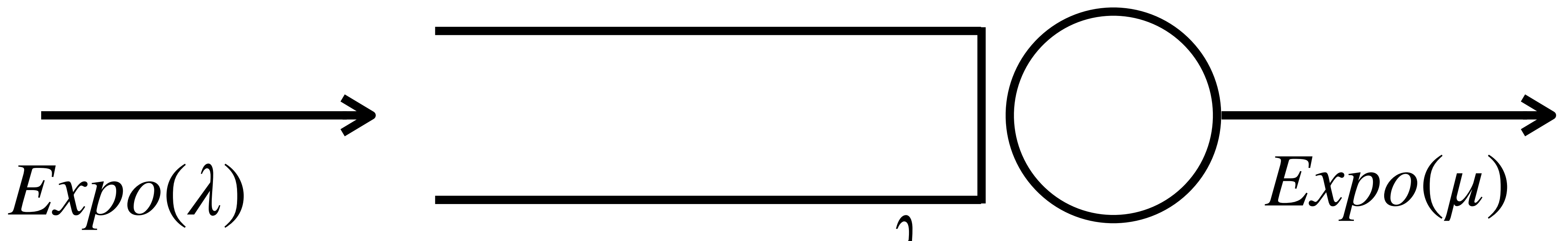

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Arrival — Departure

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Transform Method for $M/M/1$ queue

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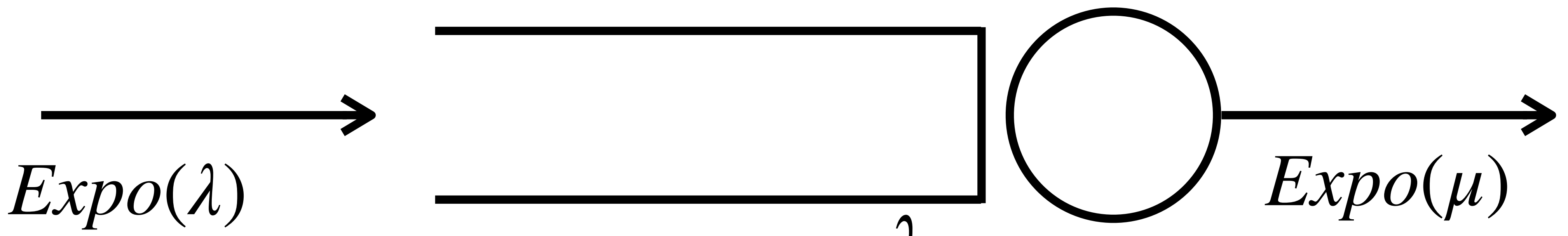

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$$= 1_{\{q=0\}}$$

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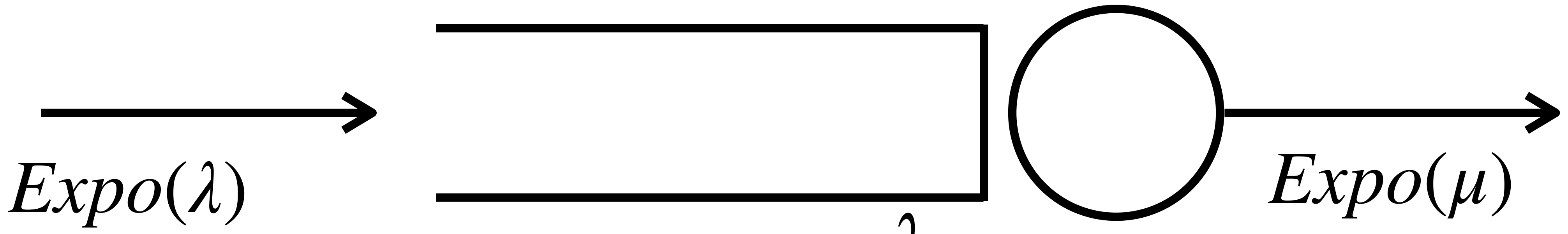
Transform Method for $M/M/1$ queue



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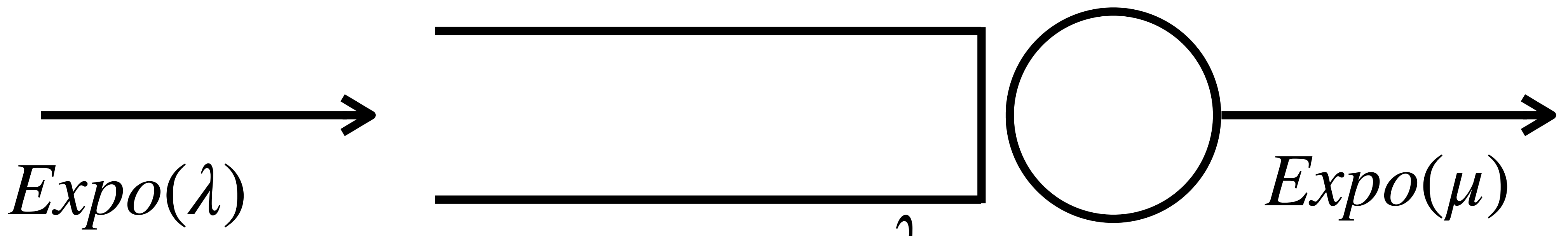
Step 2: Set drift to zero

$$\mathbb{E}\left[e^{\theta q}\right] = \frac{\mathbb{P}[q = 0]}{1 - \rho e^{\theta}}$$

Transform Method for $M/M/1$ queue



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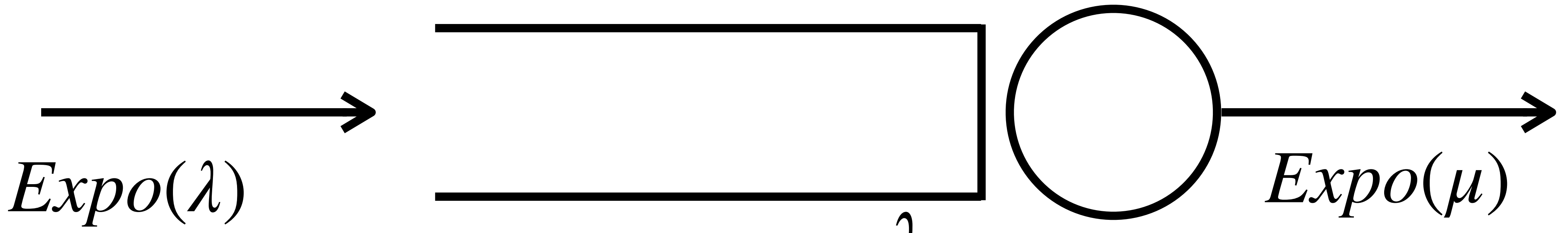

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$$e^{\theta q} 1_{\{q=0\}} = 1_{\{q=0\}}$$

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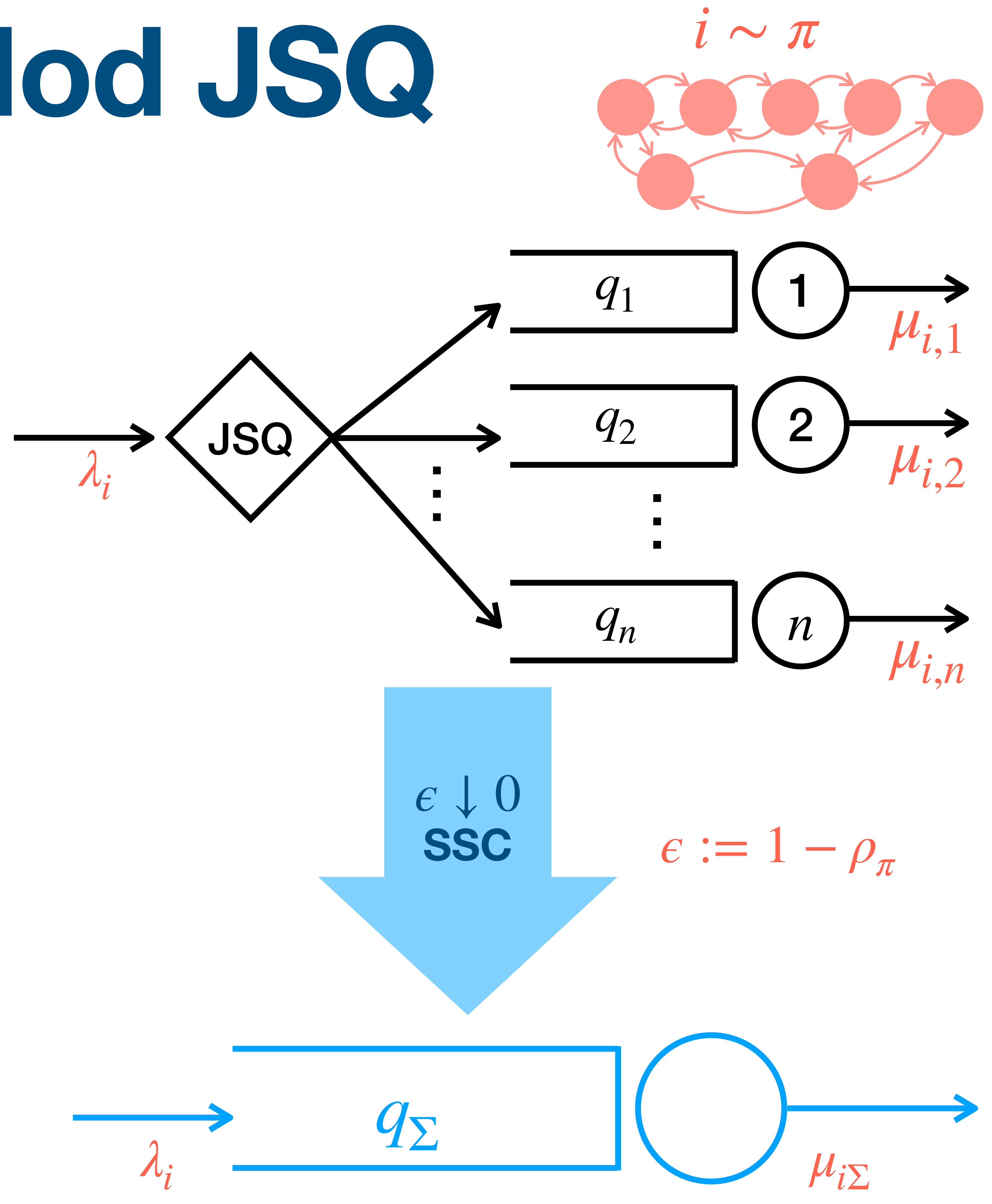
Step 3: Set $\theta = 0$

$$\mathbb{P}[q=0] = 1 - \rho$$

$$\mathbb{E}\left[e^{\theta q}\right] = \frac{1-\rho}{1-\rho e^{\theta}}$$

$\Longleftrightarrow$ $q \sim$ Geometric$(1-\rho)$

Transform Method Markov-Mod JSQ



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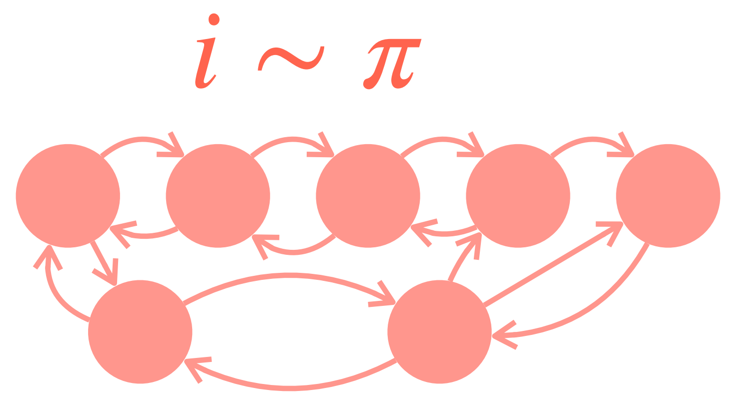
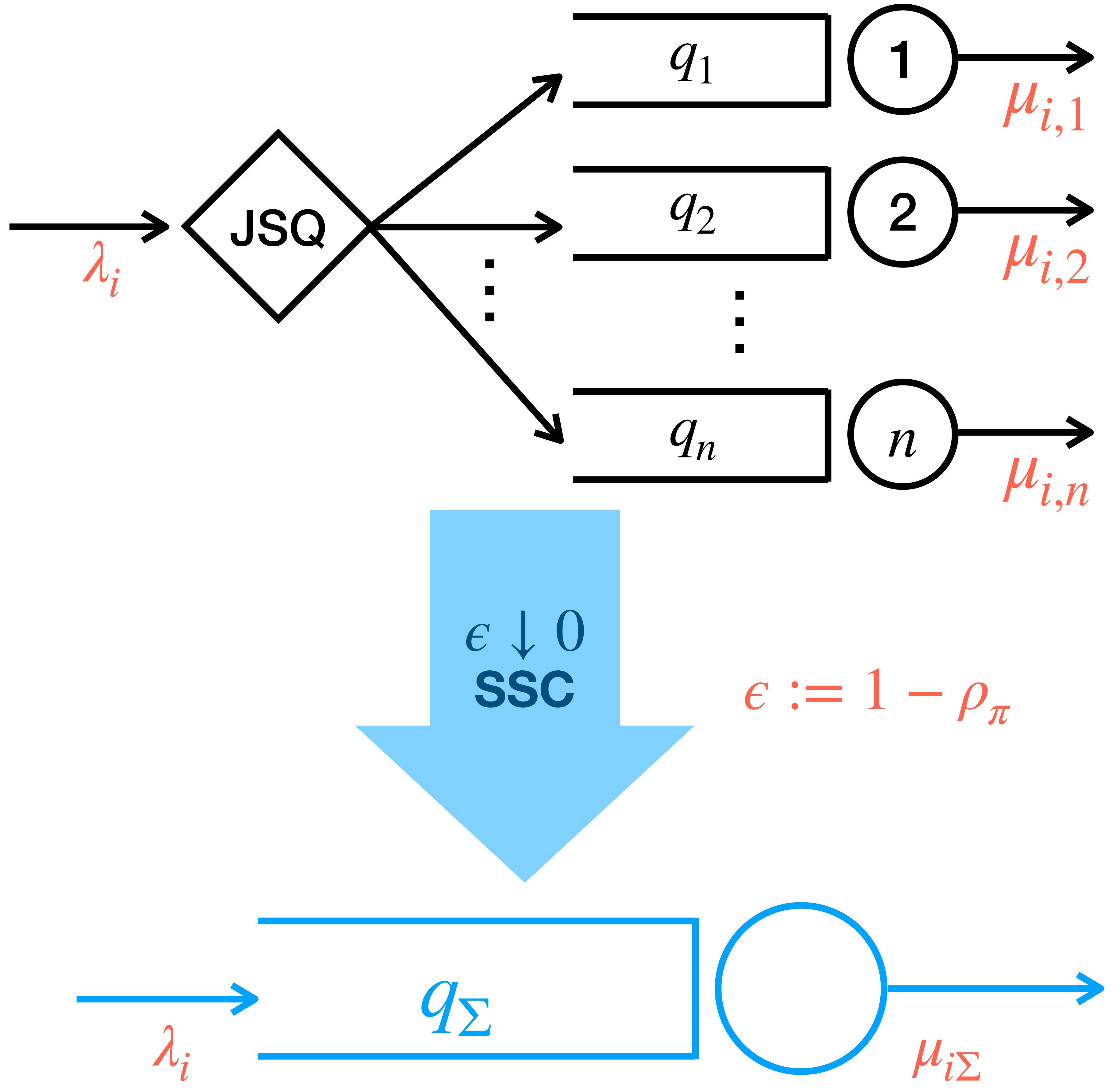


Step 1: Drift of exponential test function

$\varphi_s(i, \boldsymbol{q}) = e^{-s\epsilon q_\Sigma},\ s > 0$



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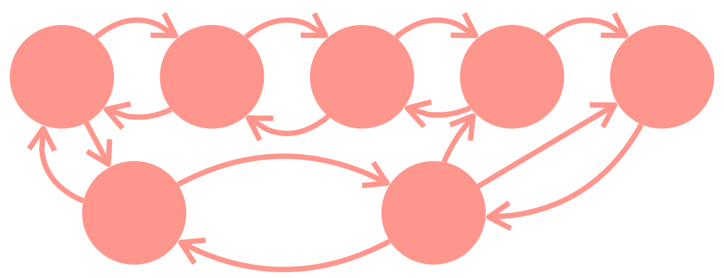

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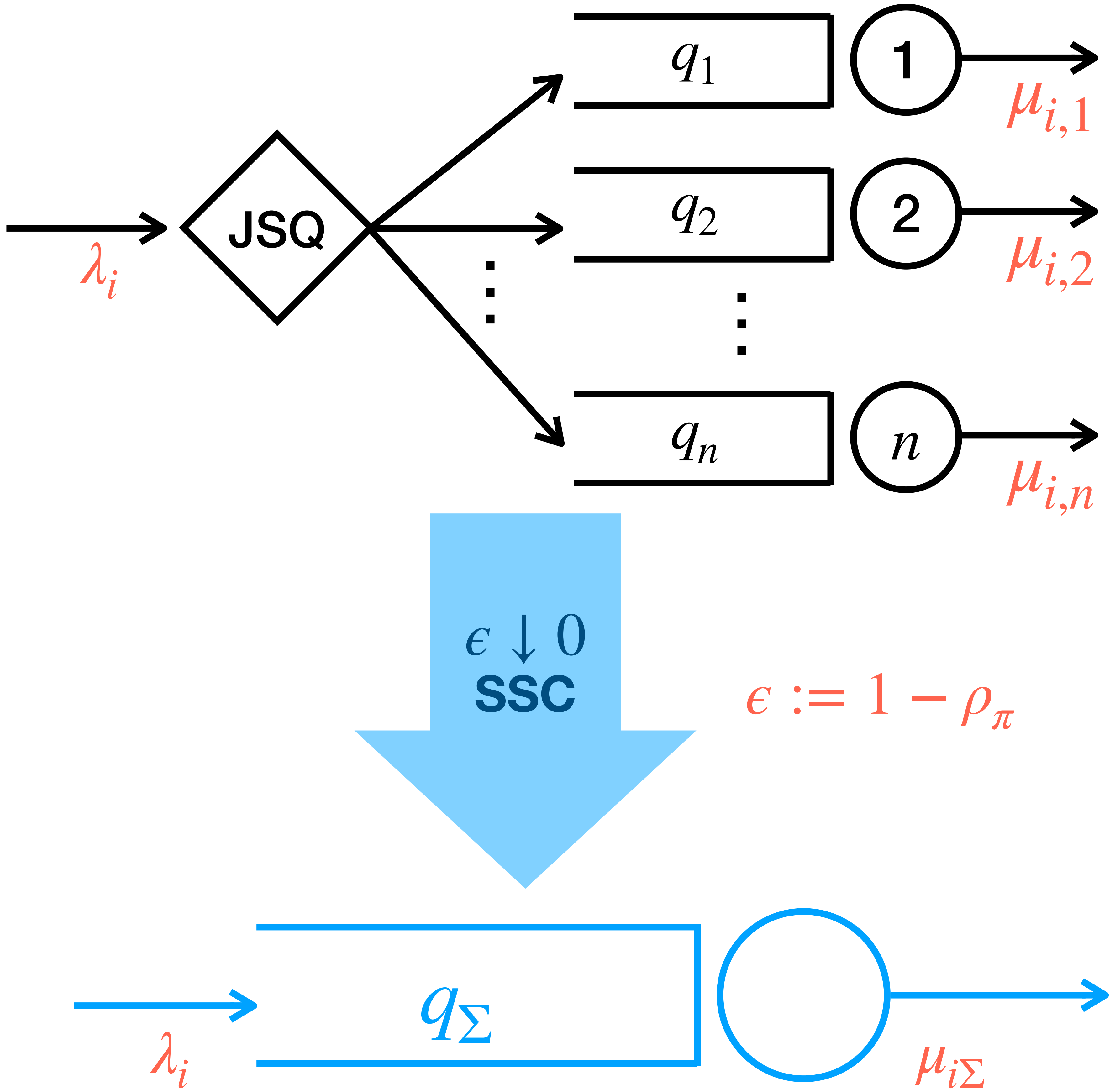

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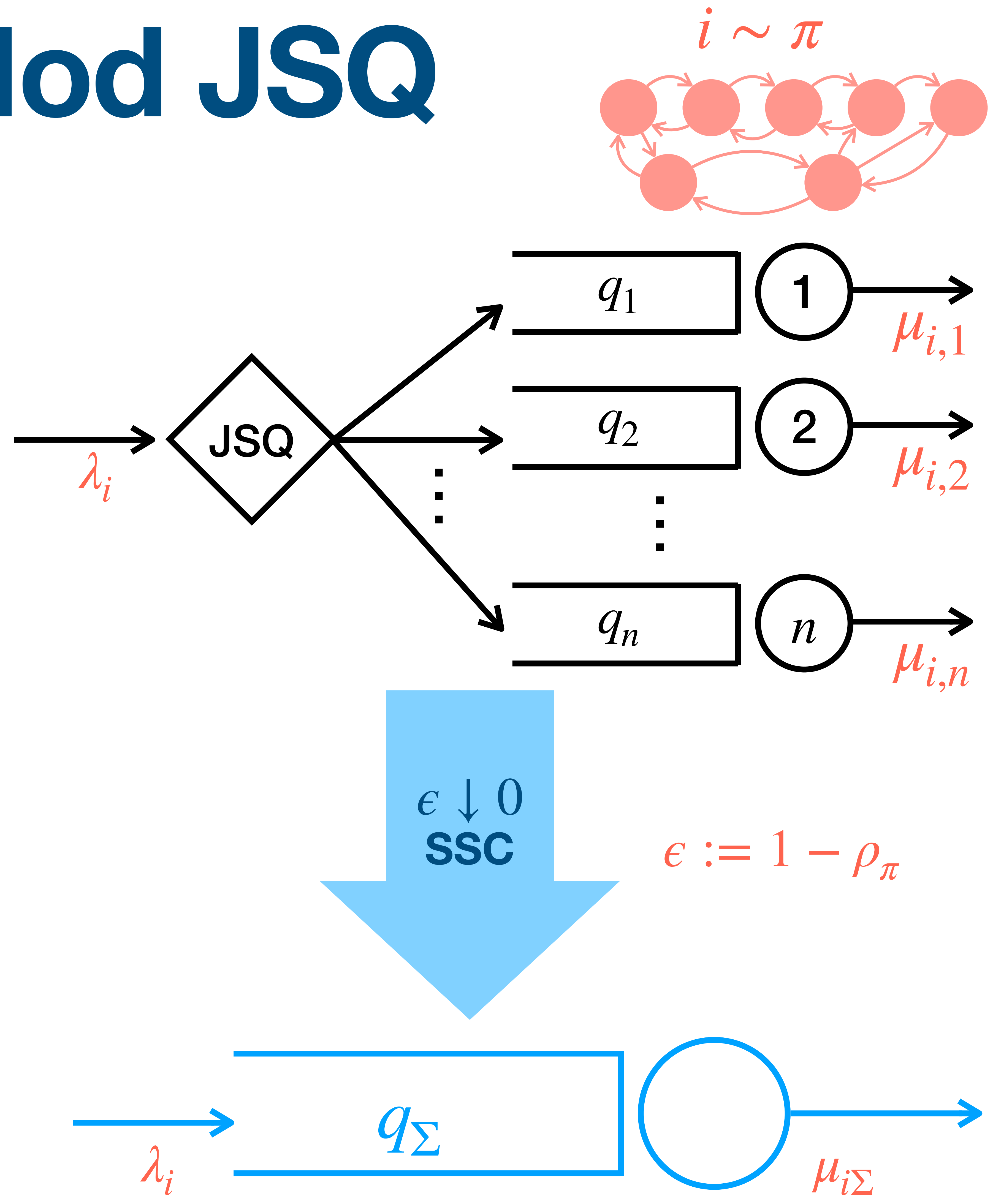
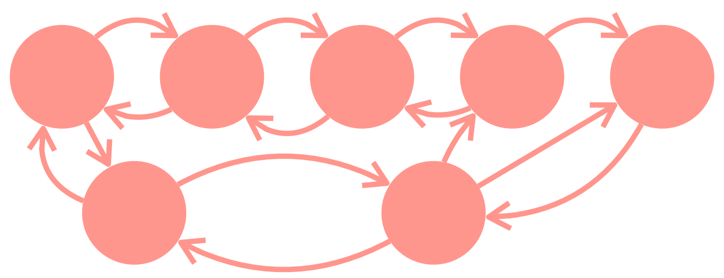


Transform Method Markov-Mod JSQ

$i \sim \pi$



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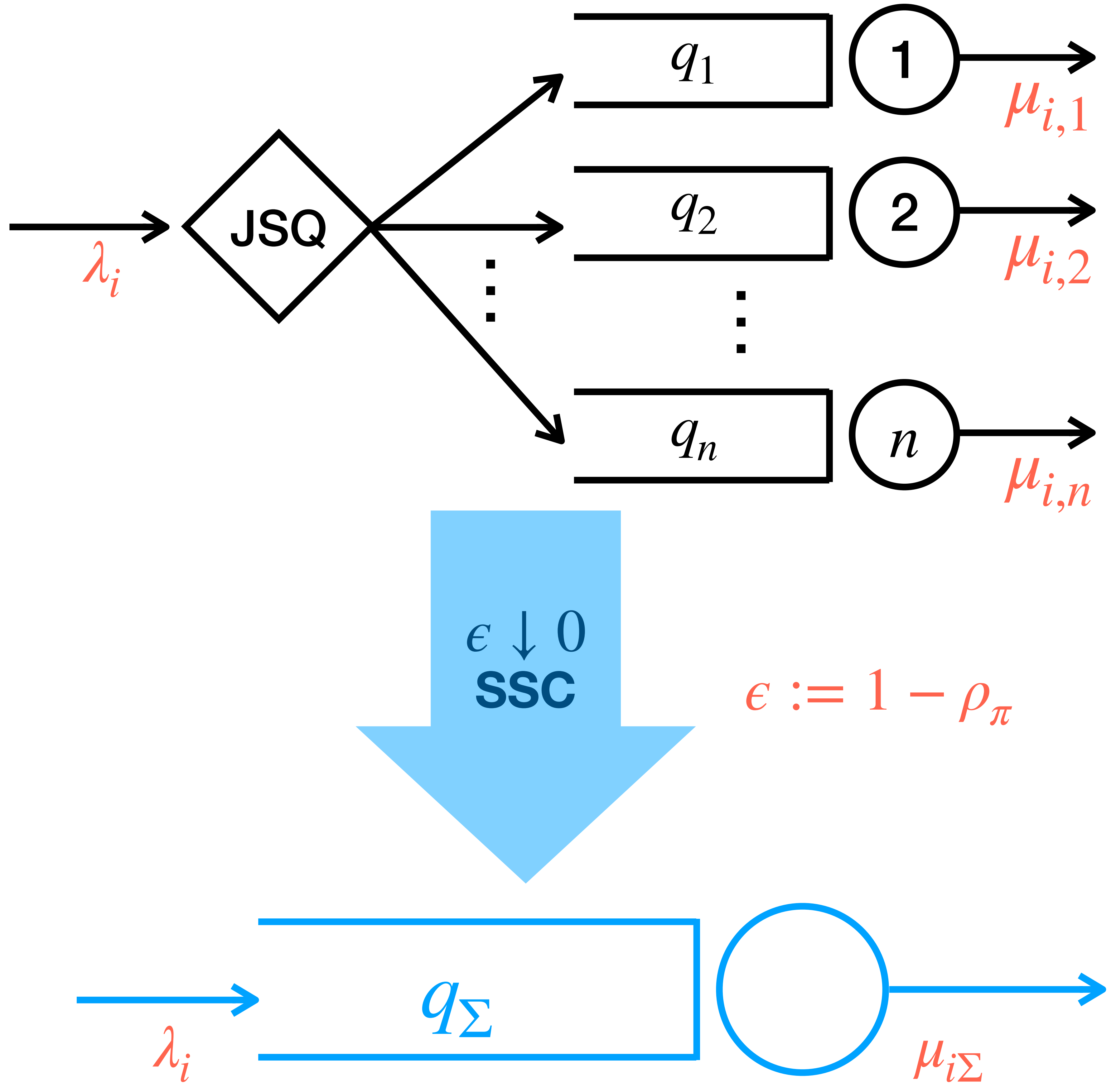
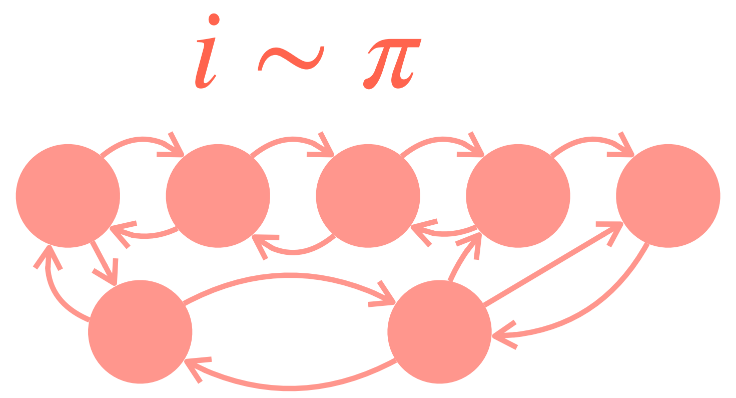
$$\approx \sum_j \mu_{ij}\, e^{-s\epsilon n q_j} 1_{\{q_j=0\}} \quad \text{(SSC)}$$



$\epsilon := 1 - \rho_\pi$

Transform Method Markov-Mod JSQ



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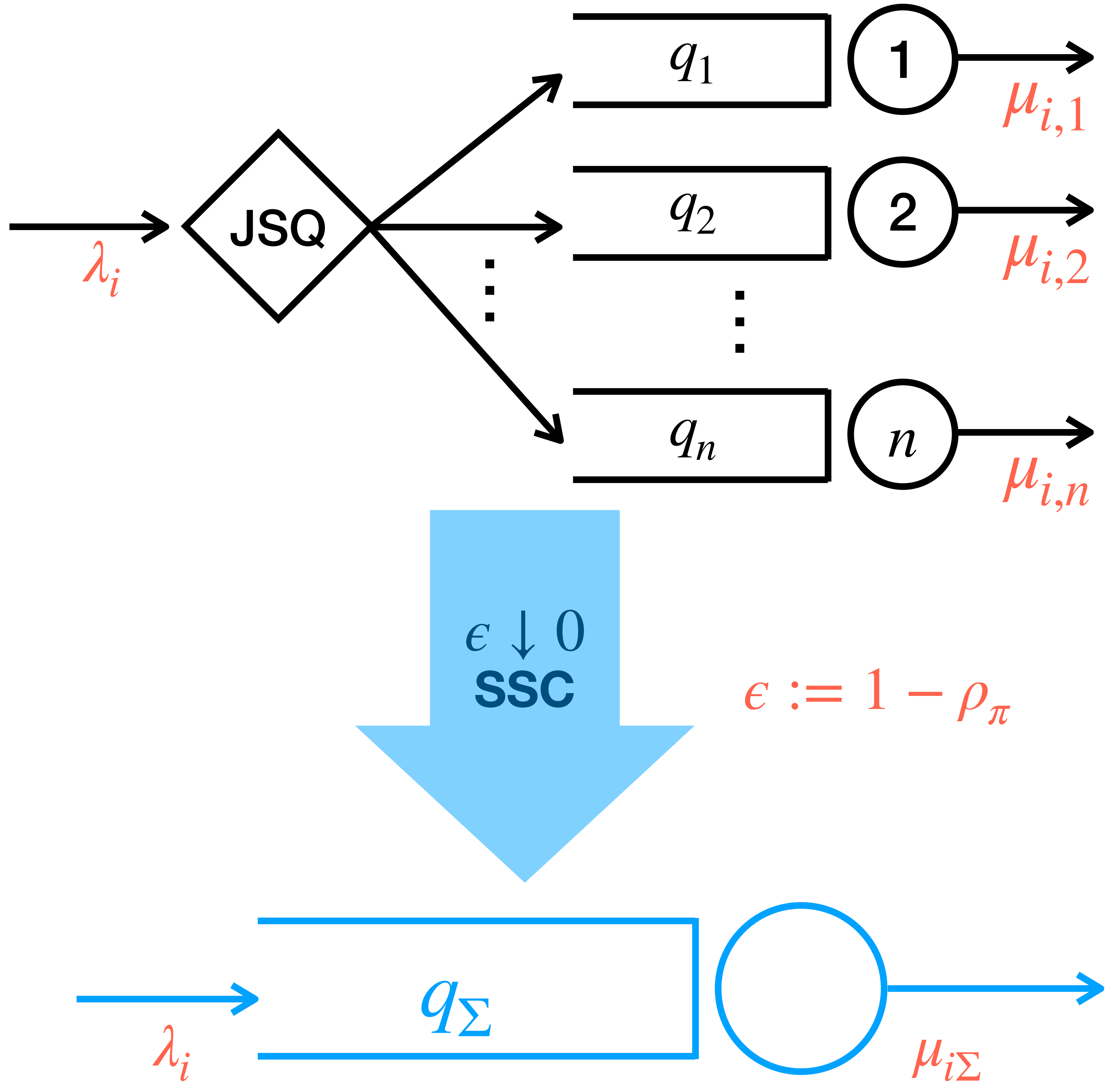

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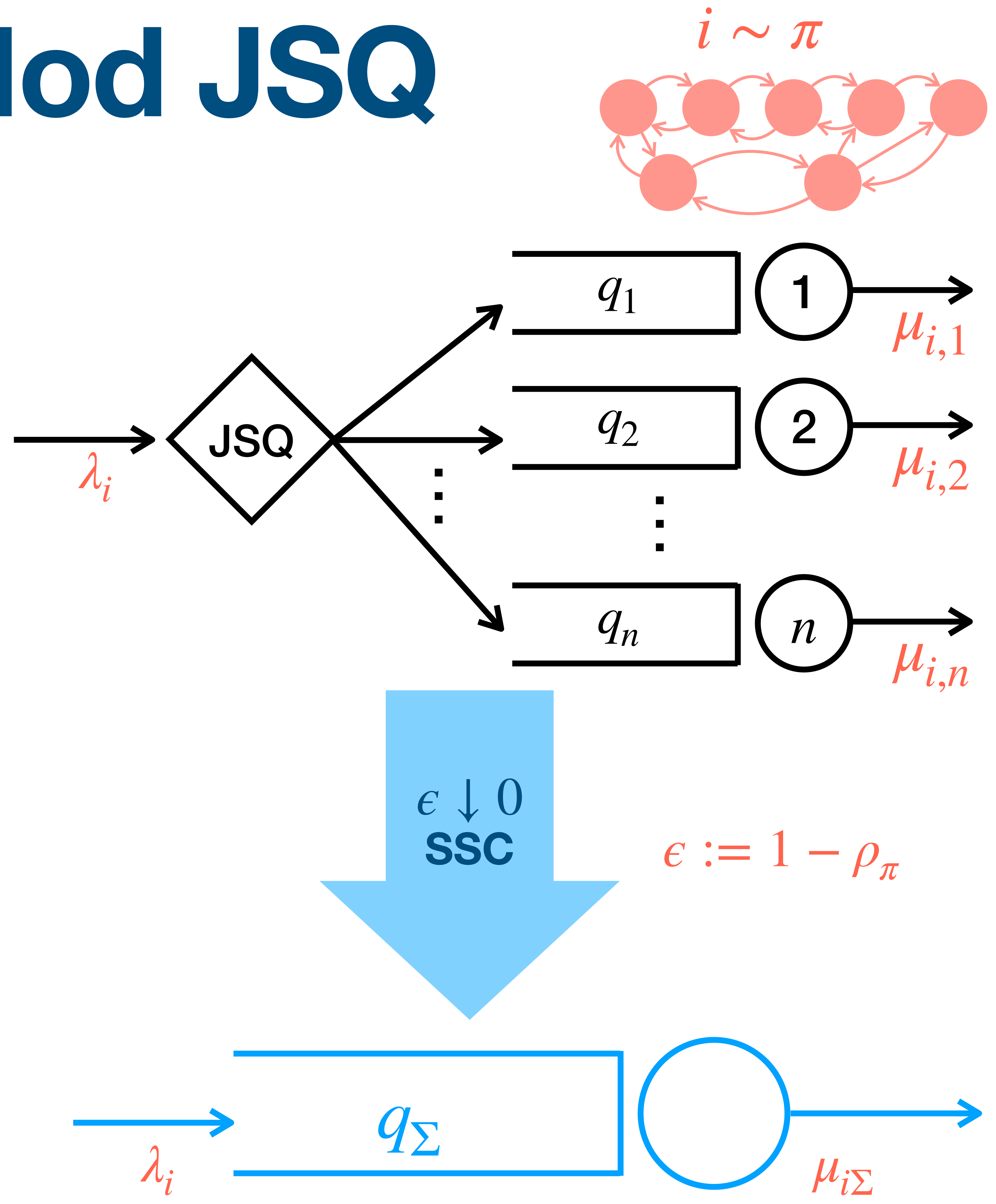

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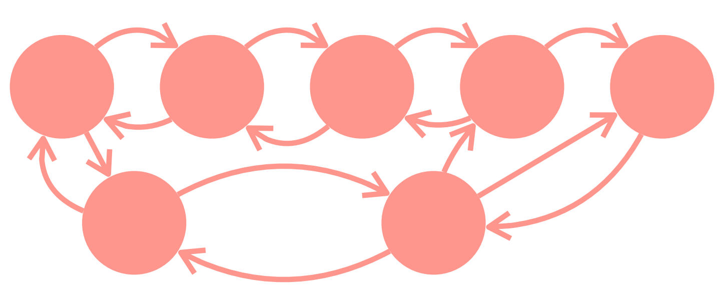

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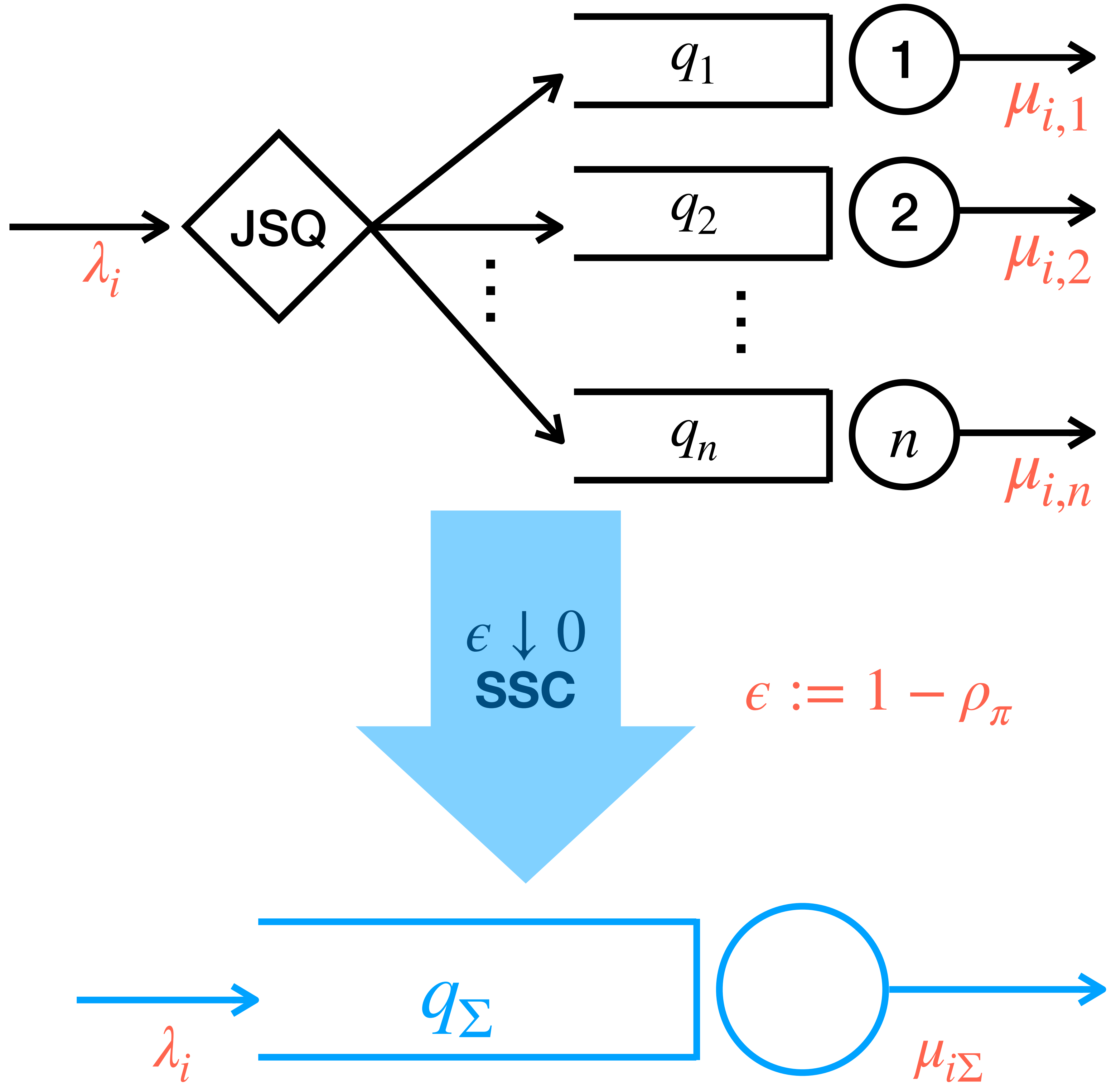

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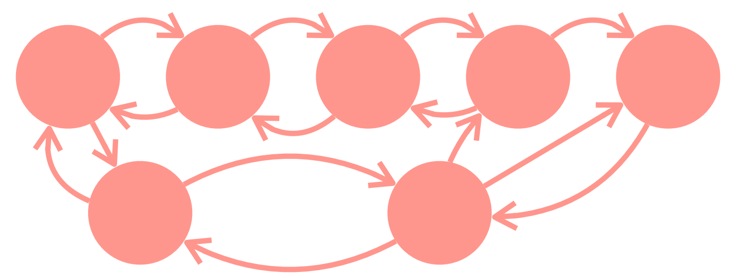

Step 1: Drift of exponential test function

$\varphi_s(i, \boldsymbol{q}) = e^{-s\epsilon q_\Sigma}$, $s > 0$

Not independent!

How to compute $\mathbb{E}[\Delta\varphi_s(i, \boldsymbol{q})]$?

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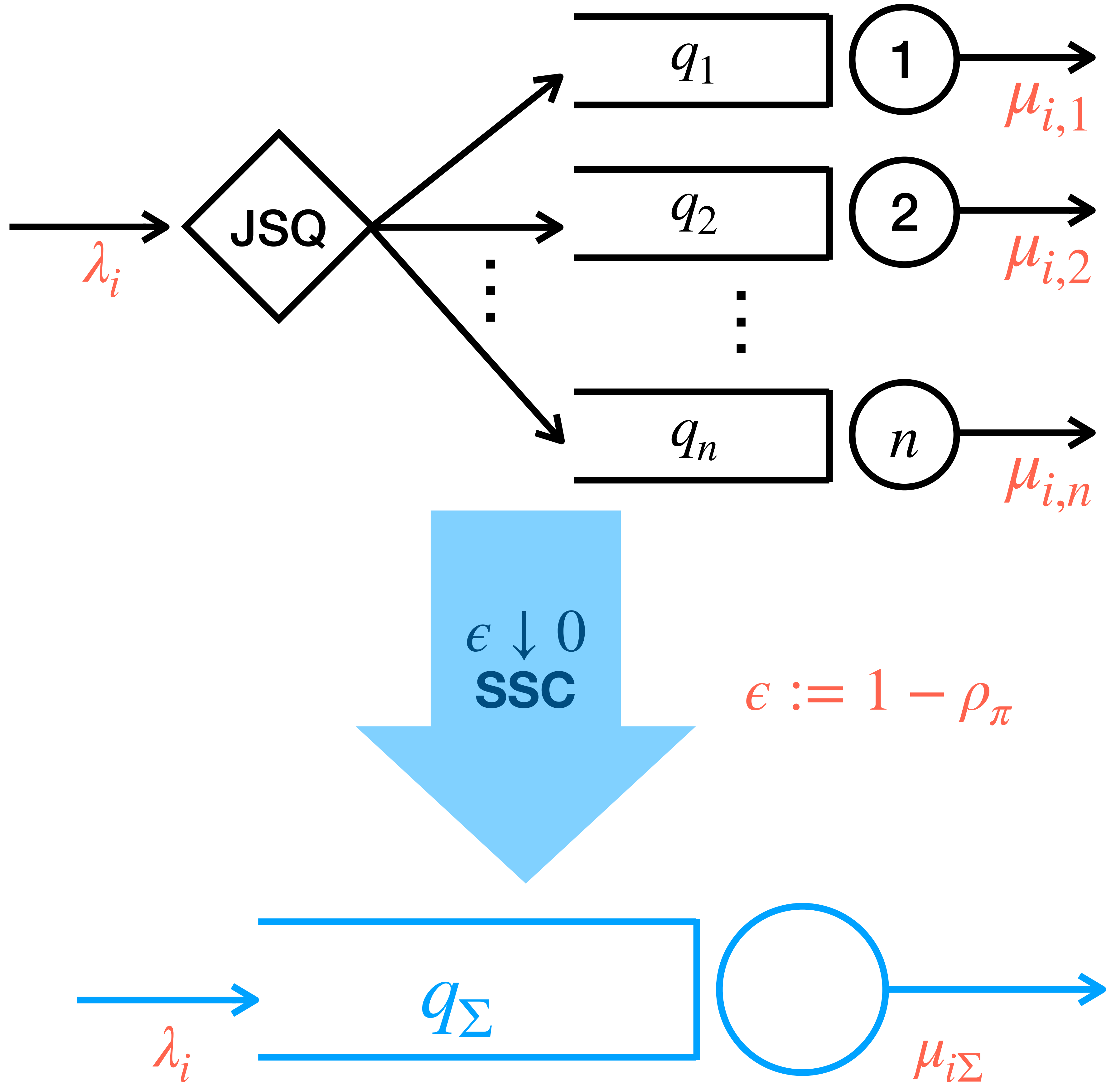
$$\mathbb{E}\Big[\sum_j \mu_{ij} 1_{\{q_j=0\}}\Big] = \mu_\Sigma\, \epsilon$$



$\epsilon := 1 - \rho_\pi$

Transform Method Markov-Mod JSQ

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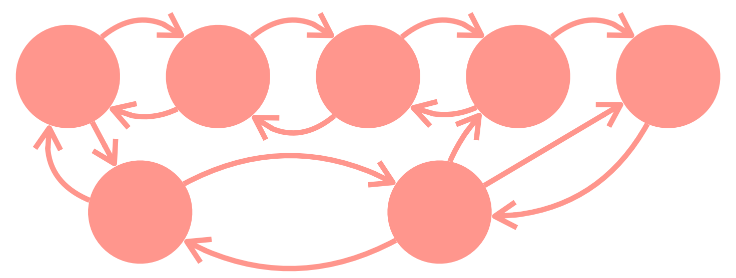

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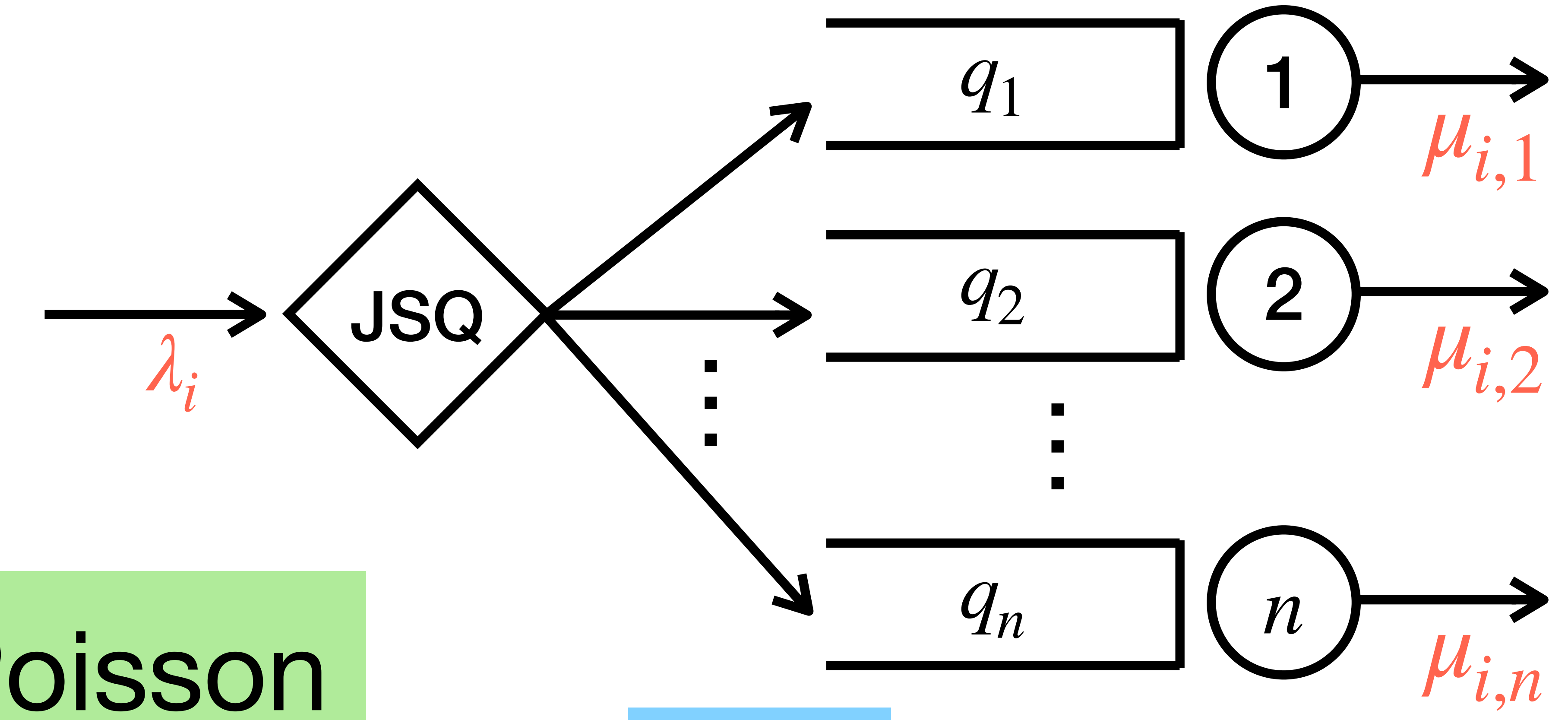
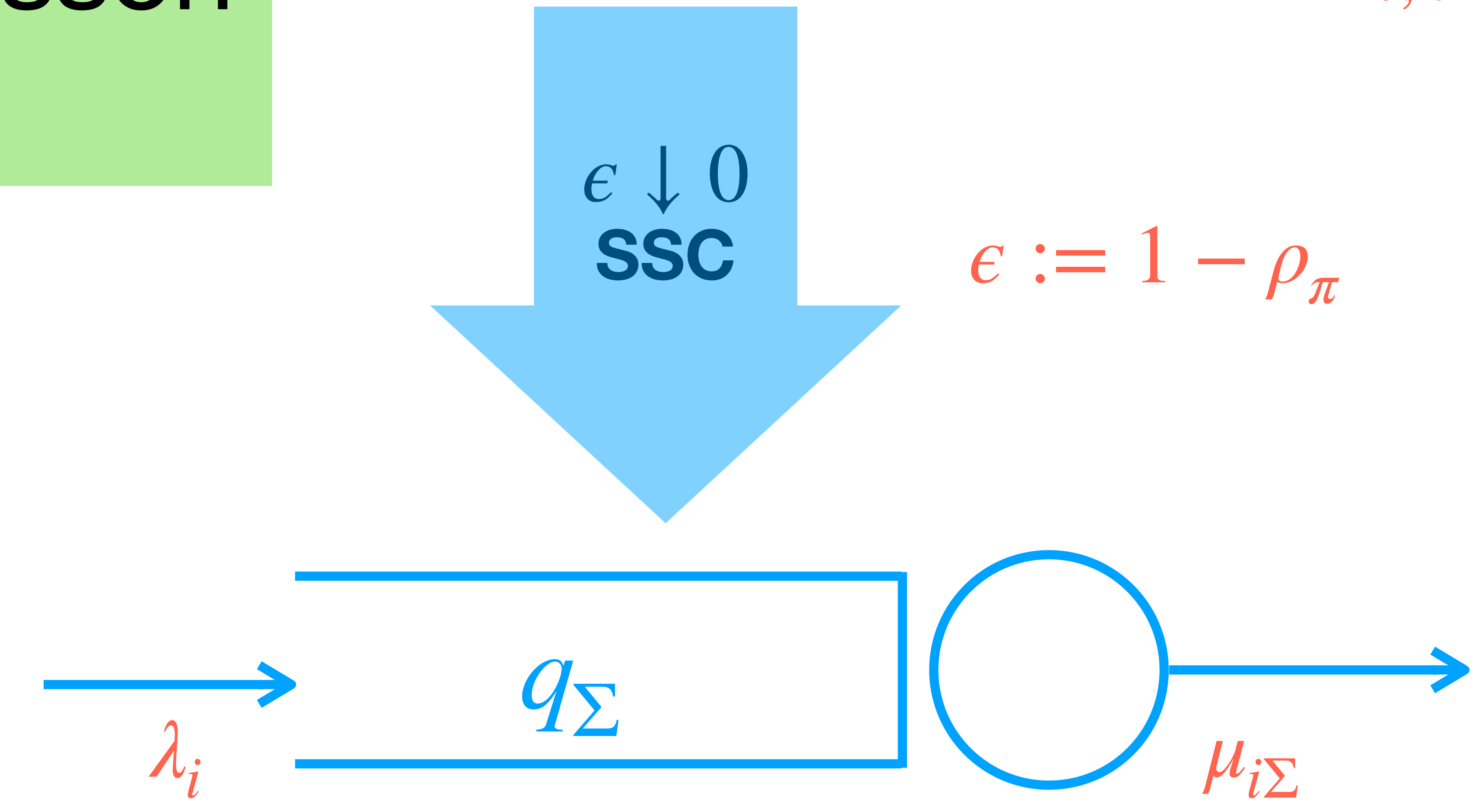
We use the Poisson equation!





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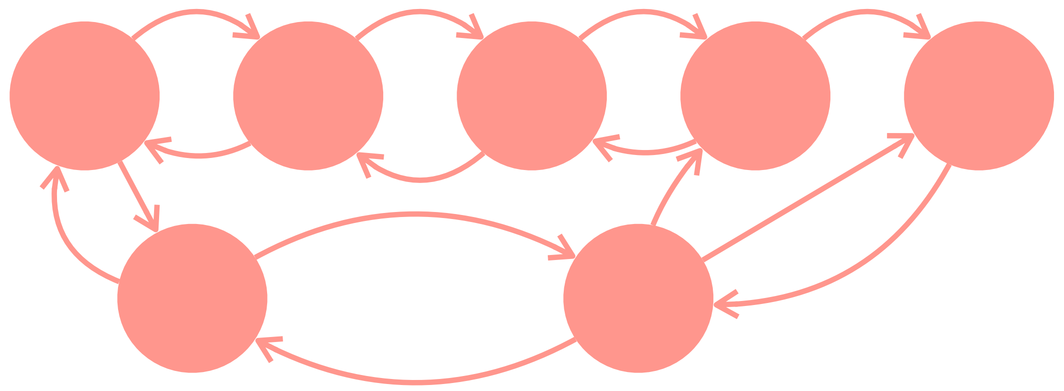
Poisson Equation for Transform Method

$Z(t) \sim$



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Let $\{Z(t)\}_t$ be a CTMC with countable state space $\mathscr{Z}$ and transition rates $\alpha_{i,i'}$
Consider a function $f : \mathscr{Z} \to \mathbb{R}$ and let $\bar{f} = \mathbb{E}[f(Z)]$. Then, there exists a function $V_f : \mathscr{Z} \to \mathbb{R}$ such that

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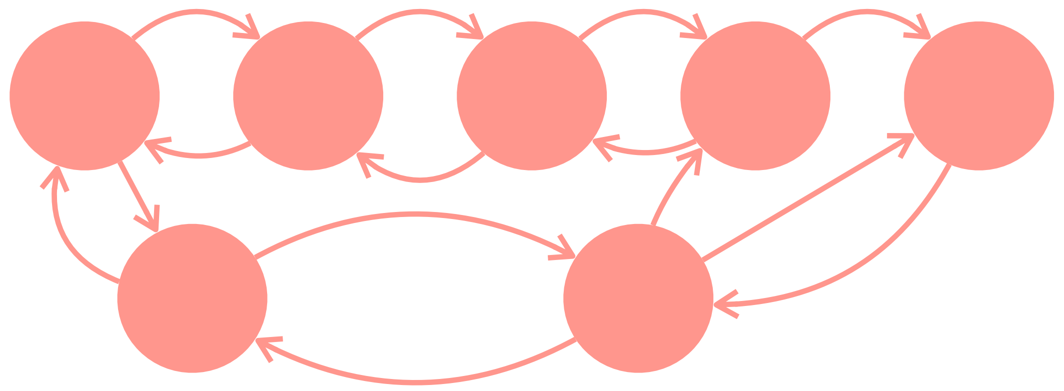

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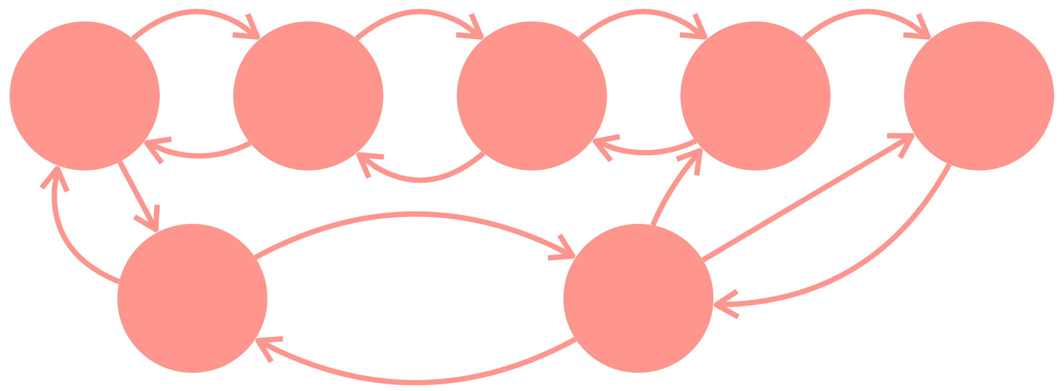

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For any function $f : \mathcal{Z} \to \mathbb{R}$ such that $V_f(i)$ exists and $\mathbb{E}\left[\,|V_f(i)|^{1+\eta}\right] \leq C$ for some $\eta > 0$,

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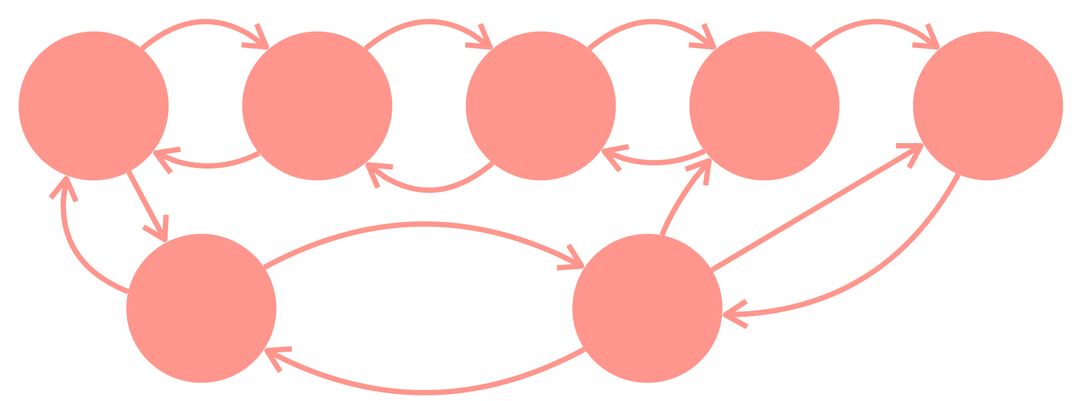

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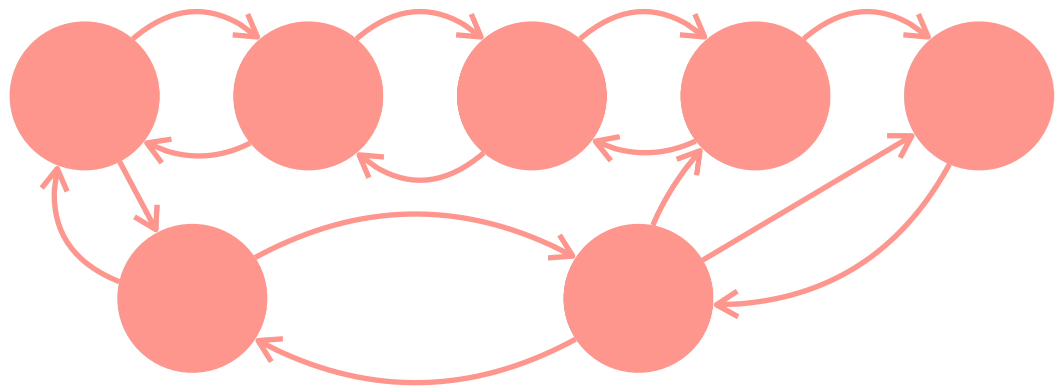

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Solution to
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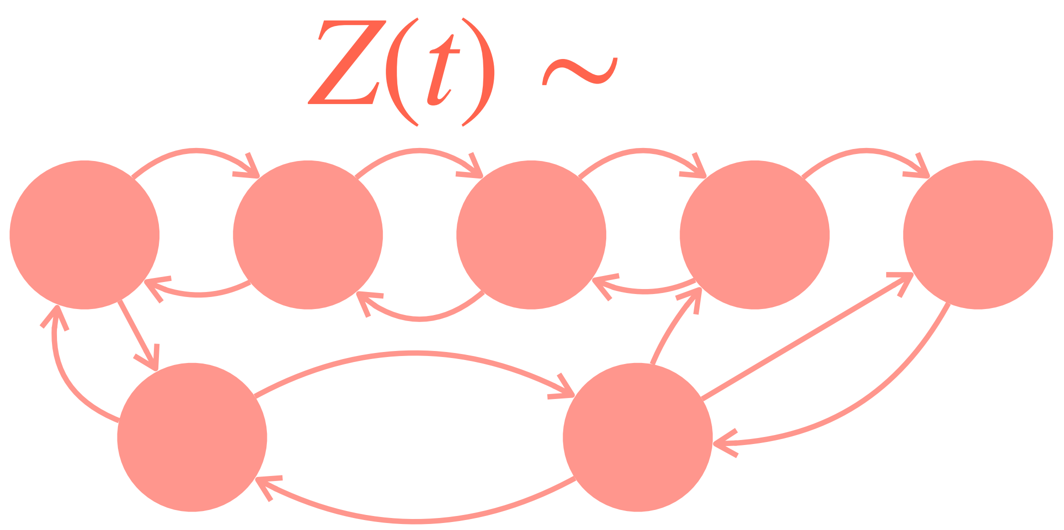
Poisson Equation for Transform Method



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Solution to Poisson equation

Error term

Applying the Poisson Equation

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Then, put everything together. ■

Key Takeaways

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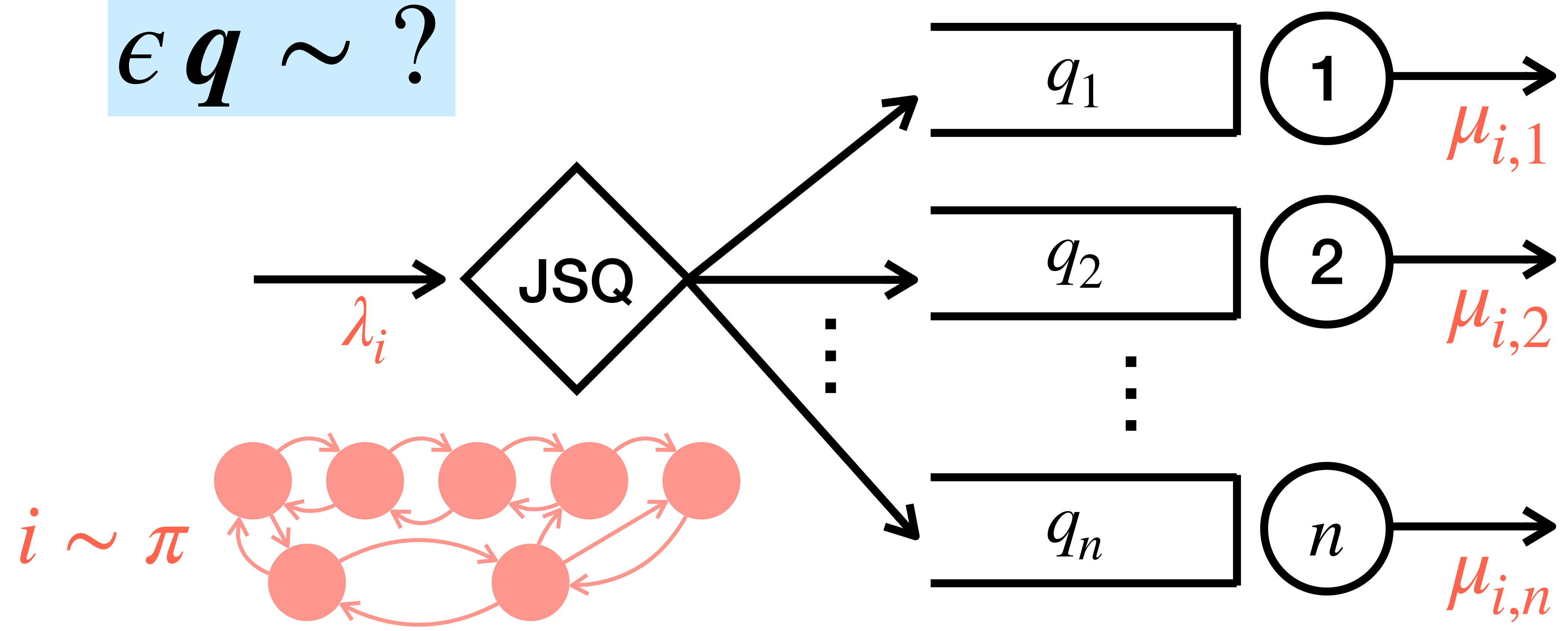
$\epsilon \boldsymbol{q} \sim ?$



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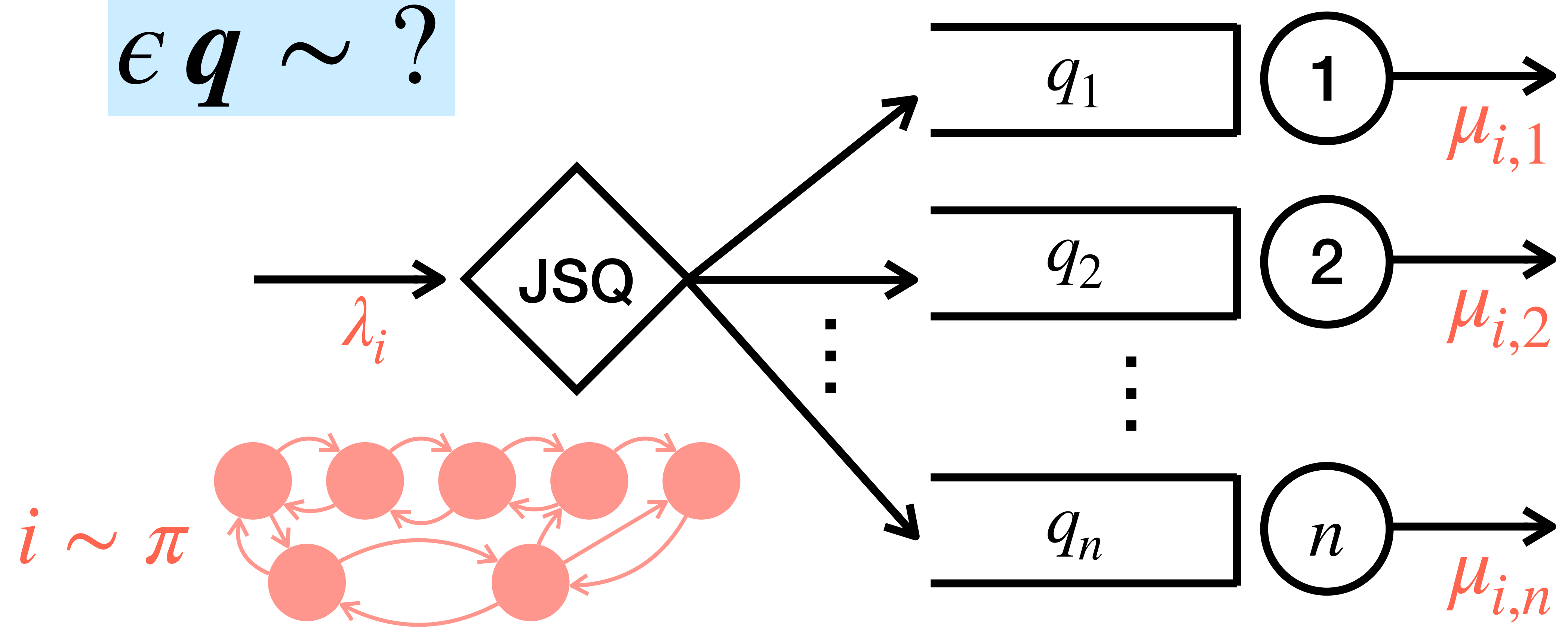

Step 1: State Space Collapse

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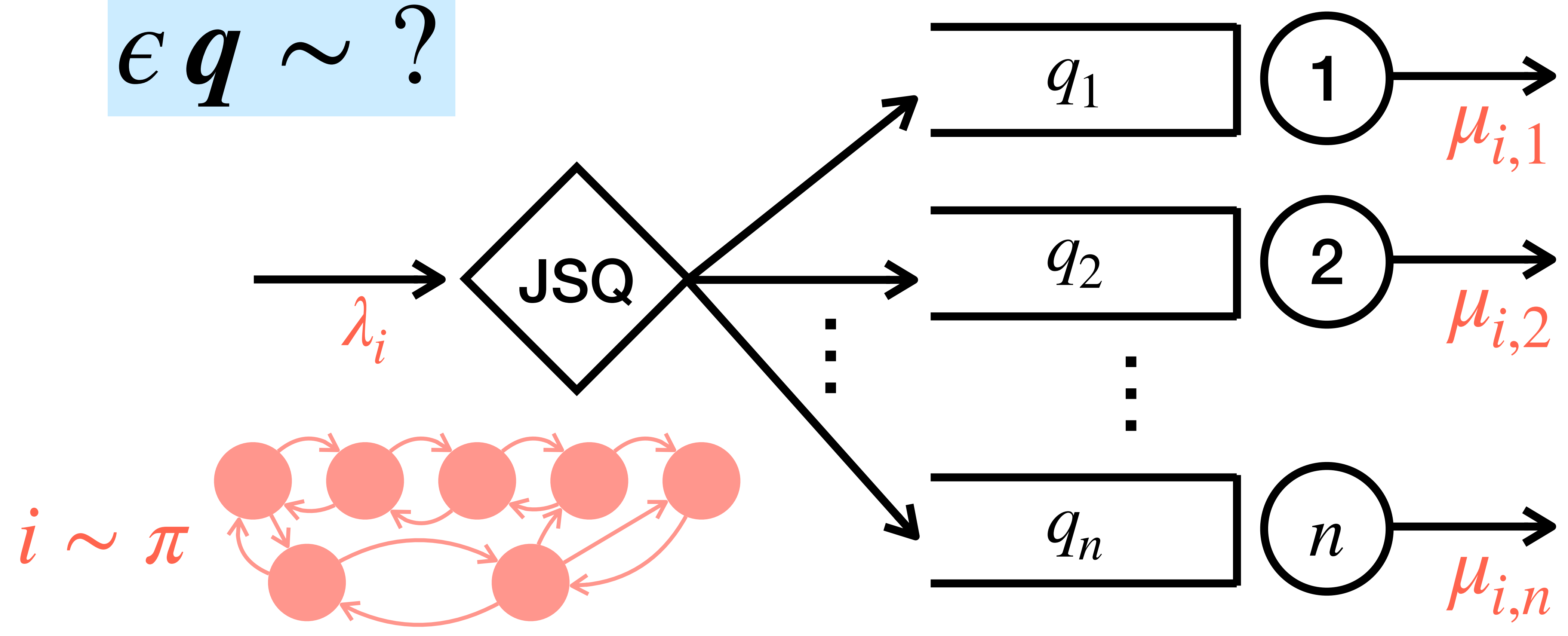

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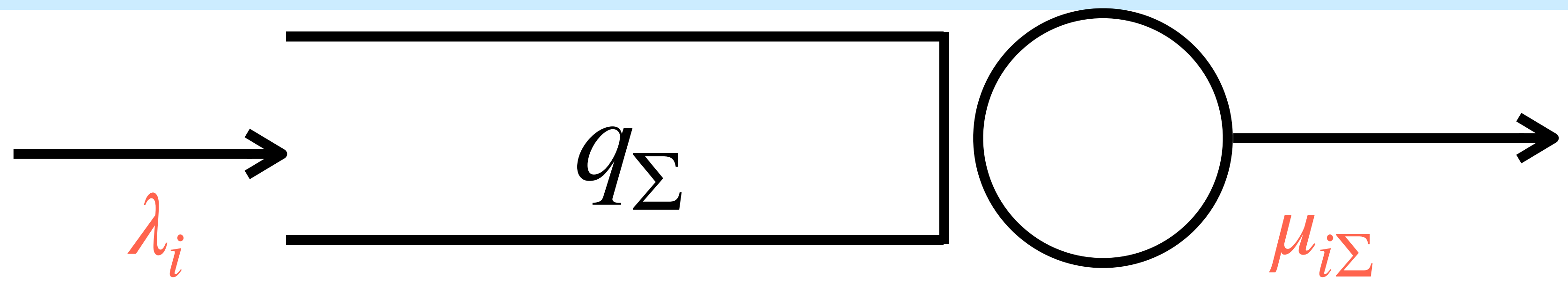

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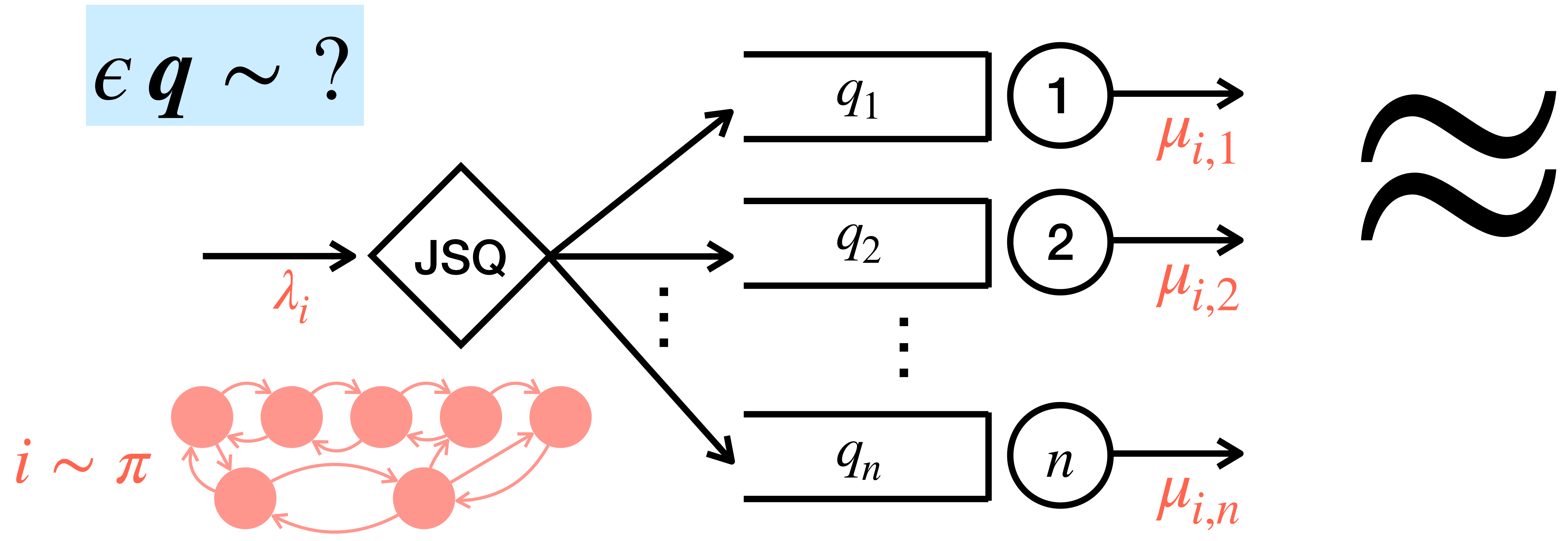

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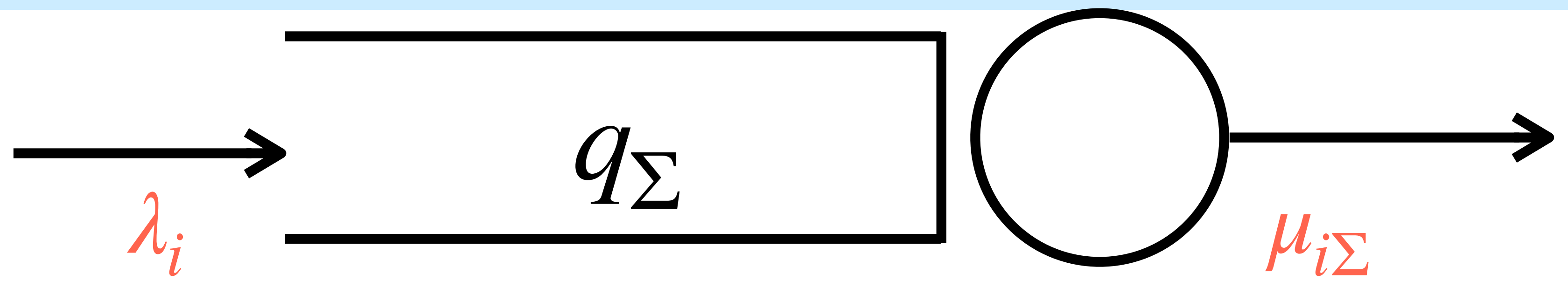

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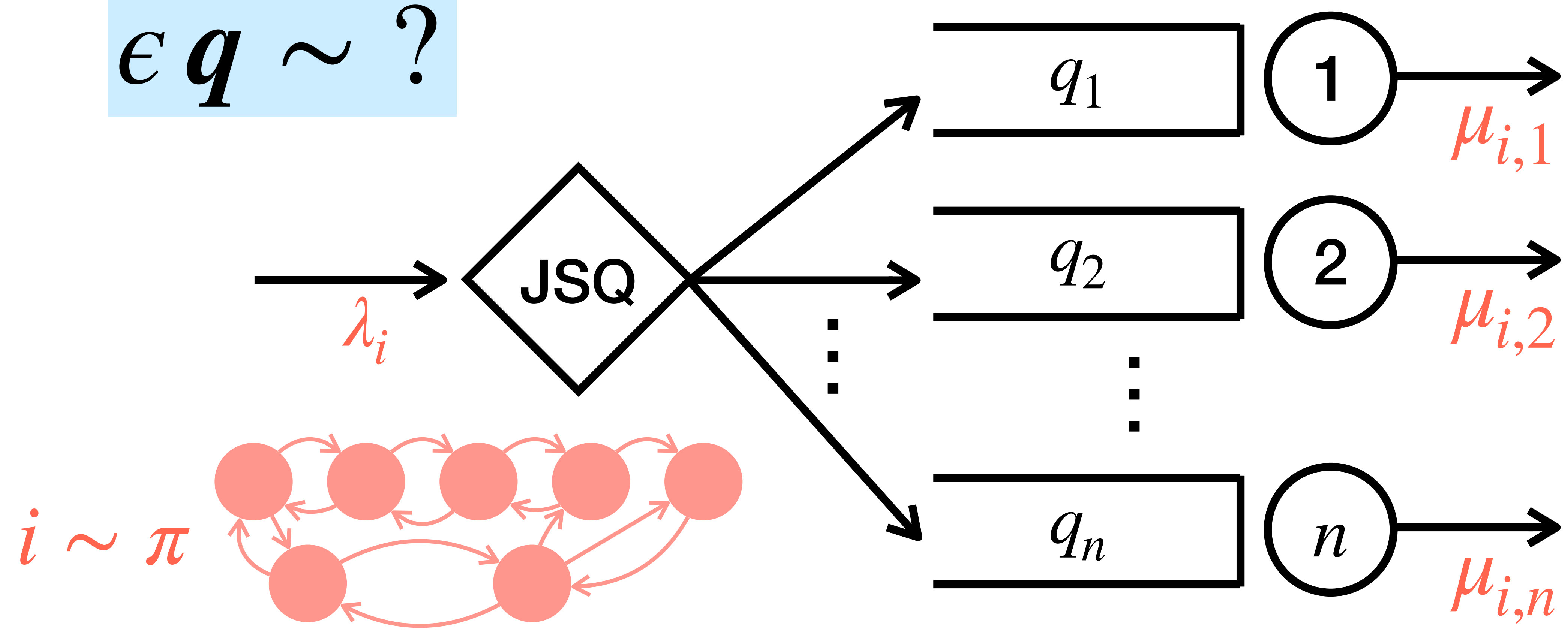

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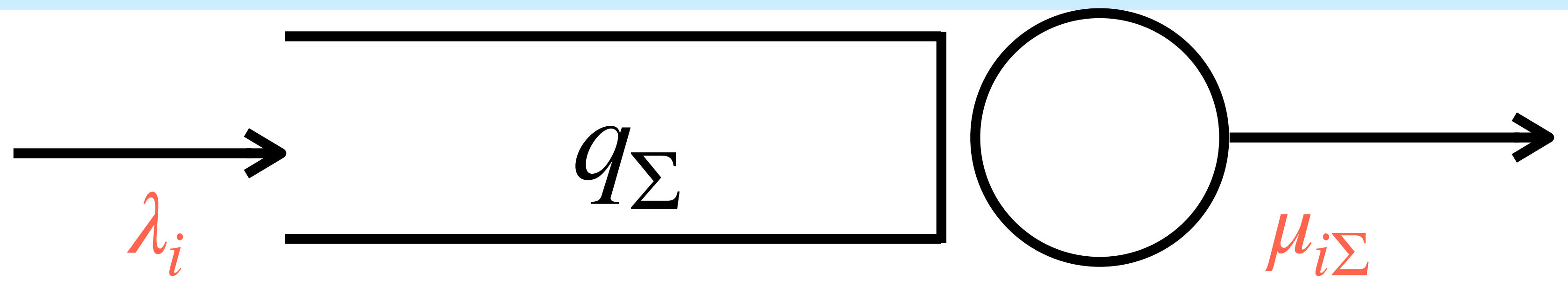

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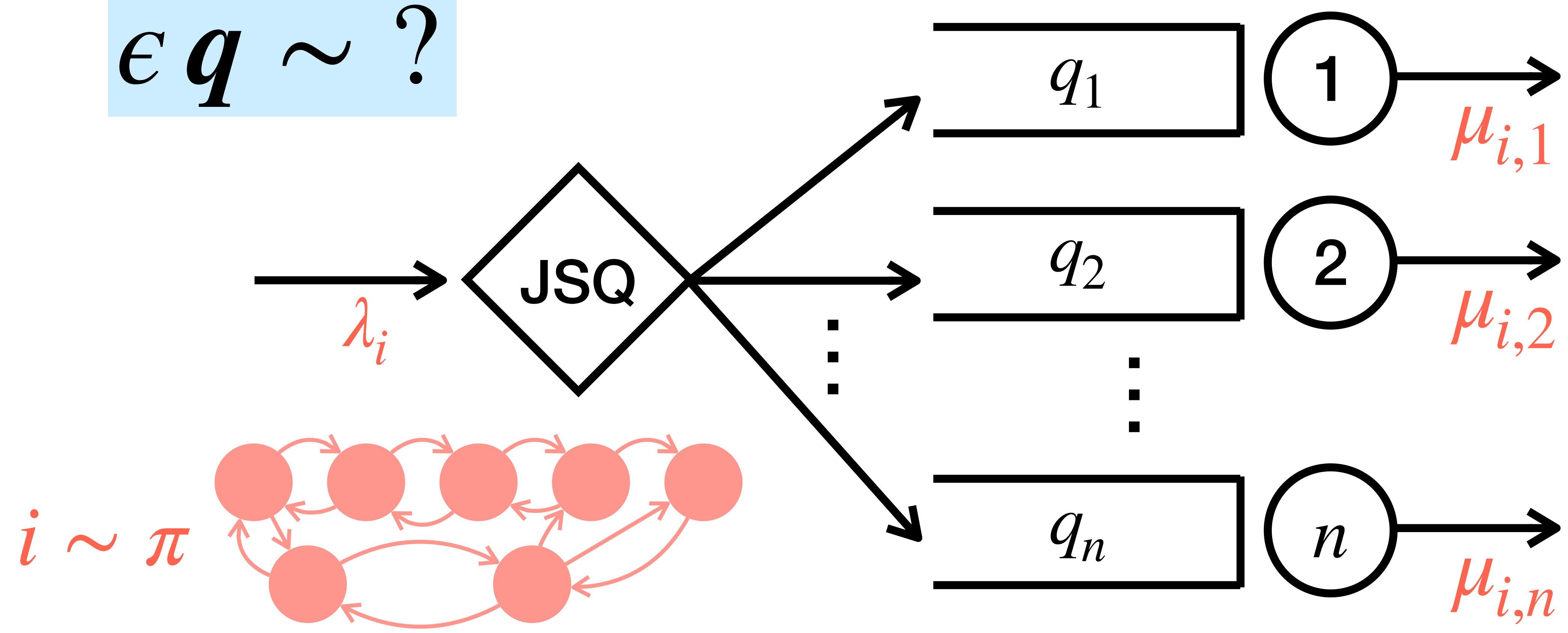

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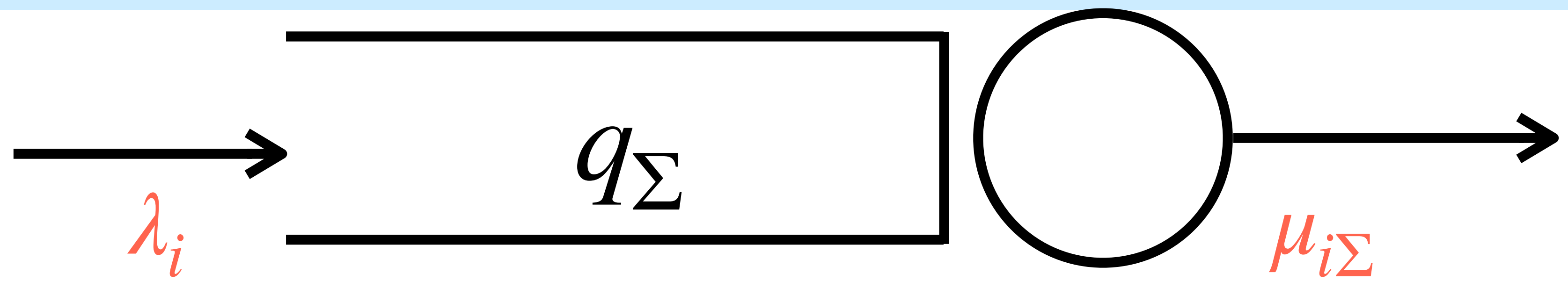

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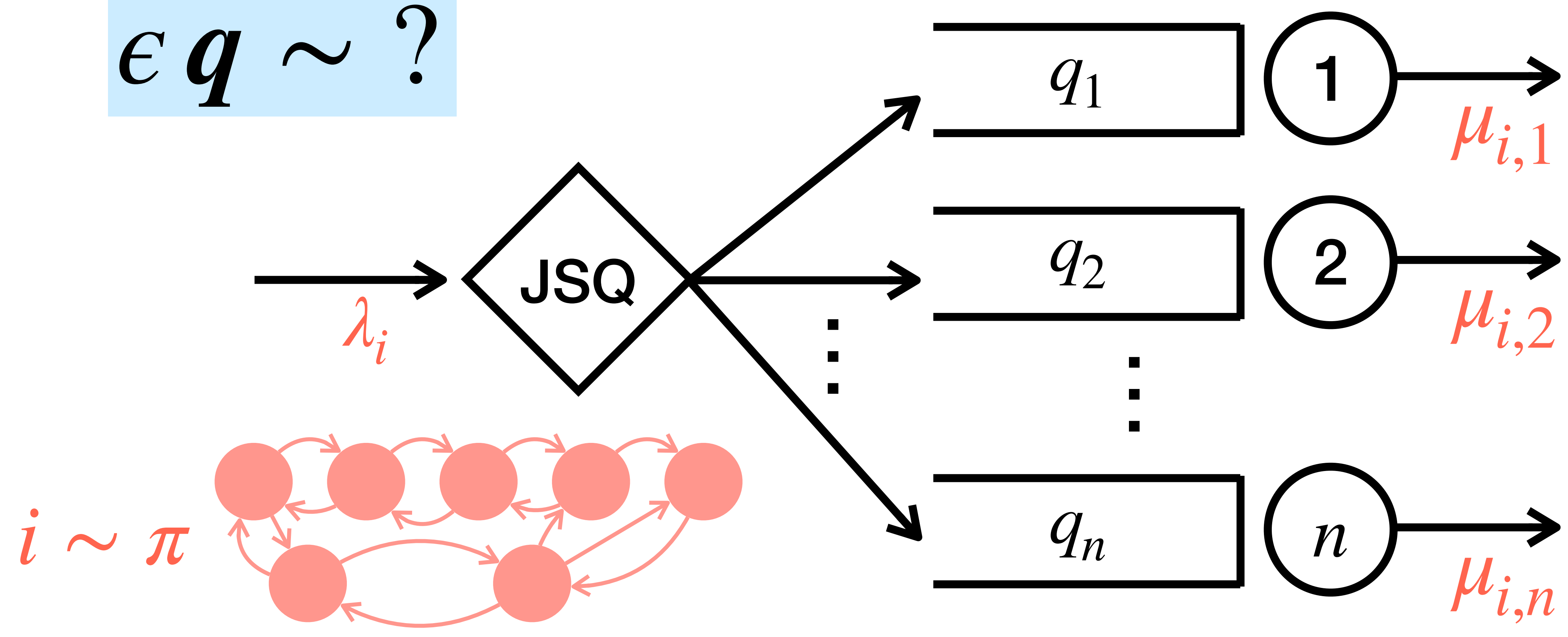

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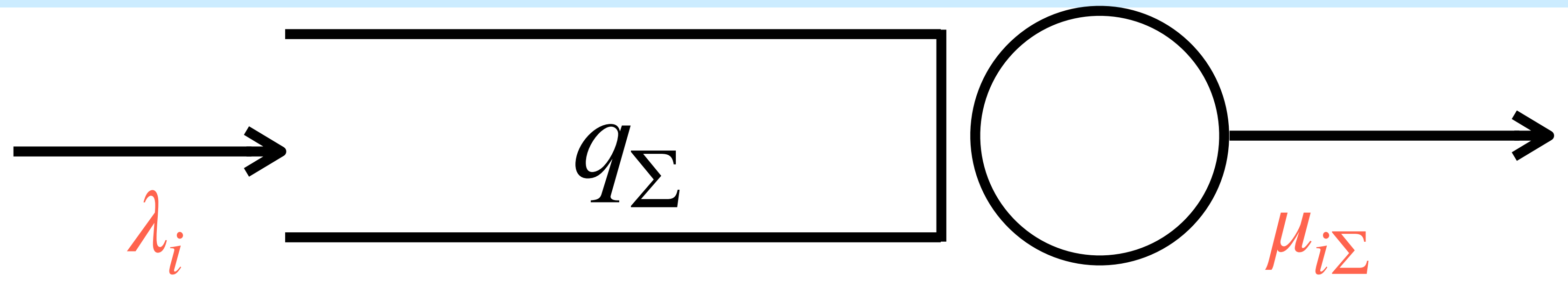

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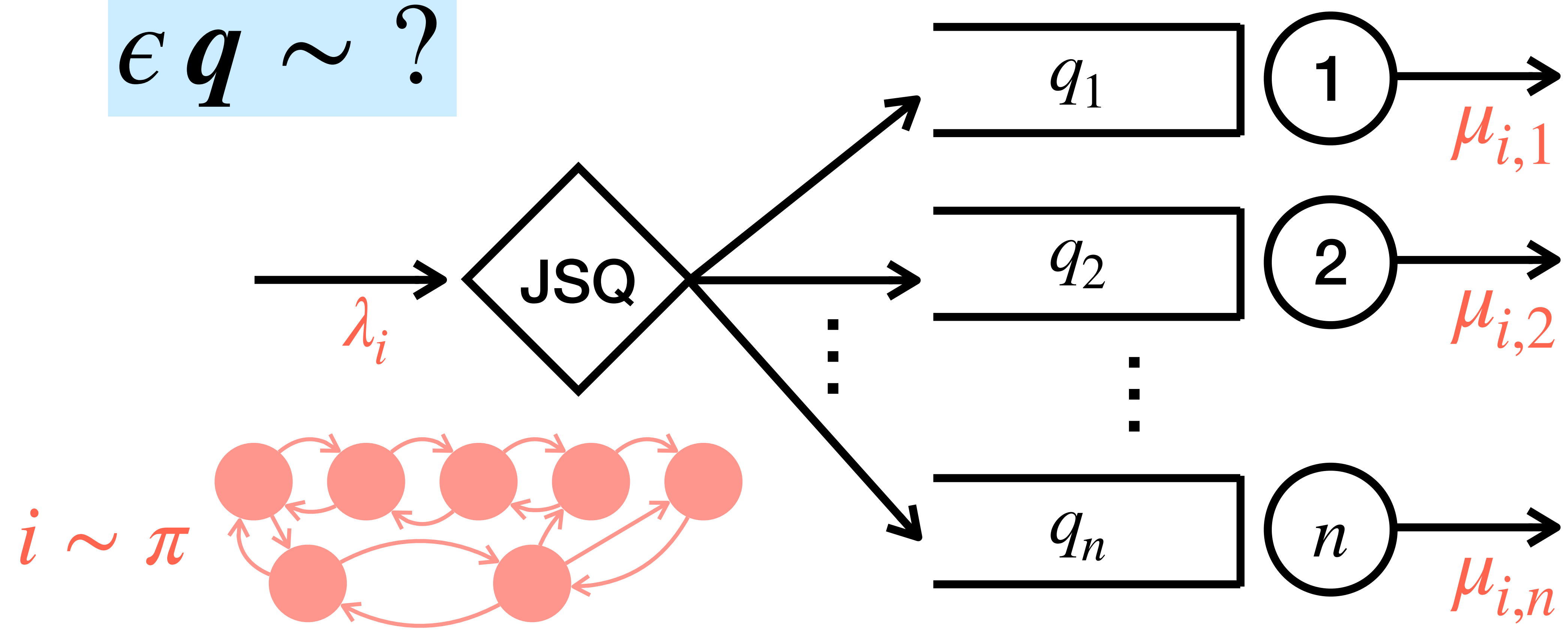

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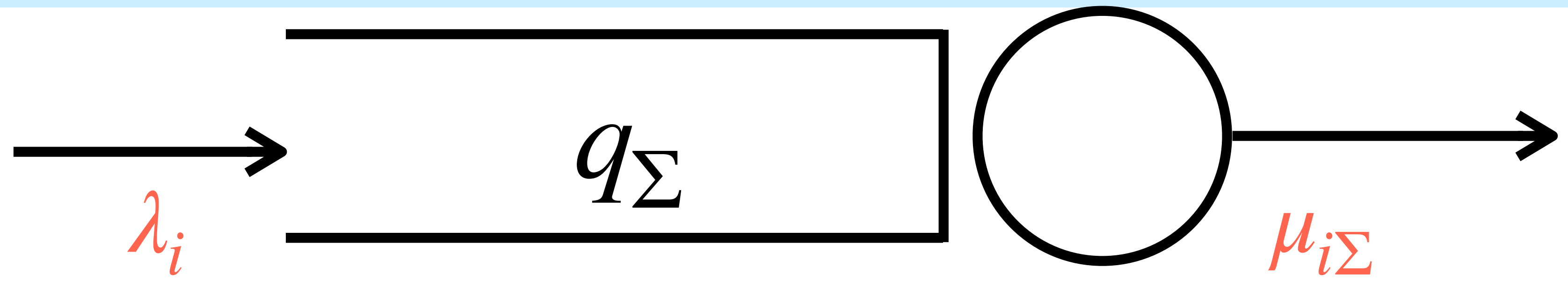

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